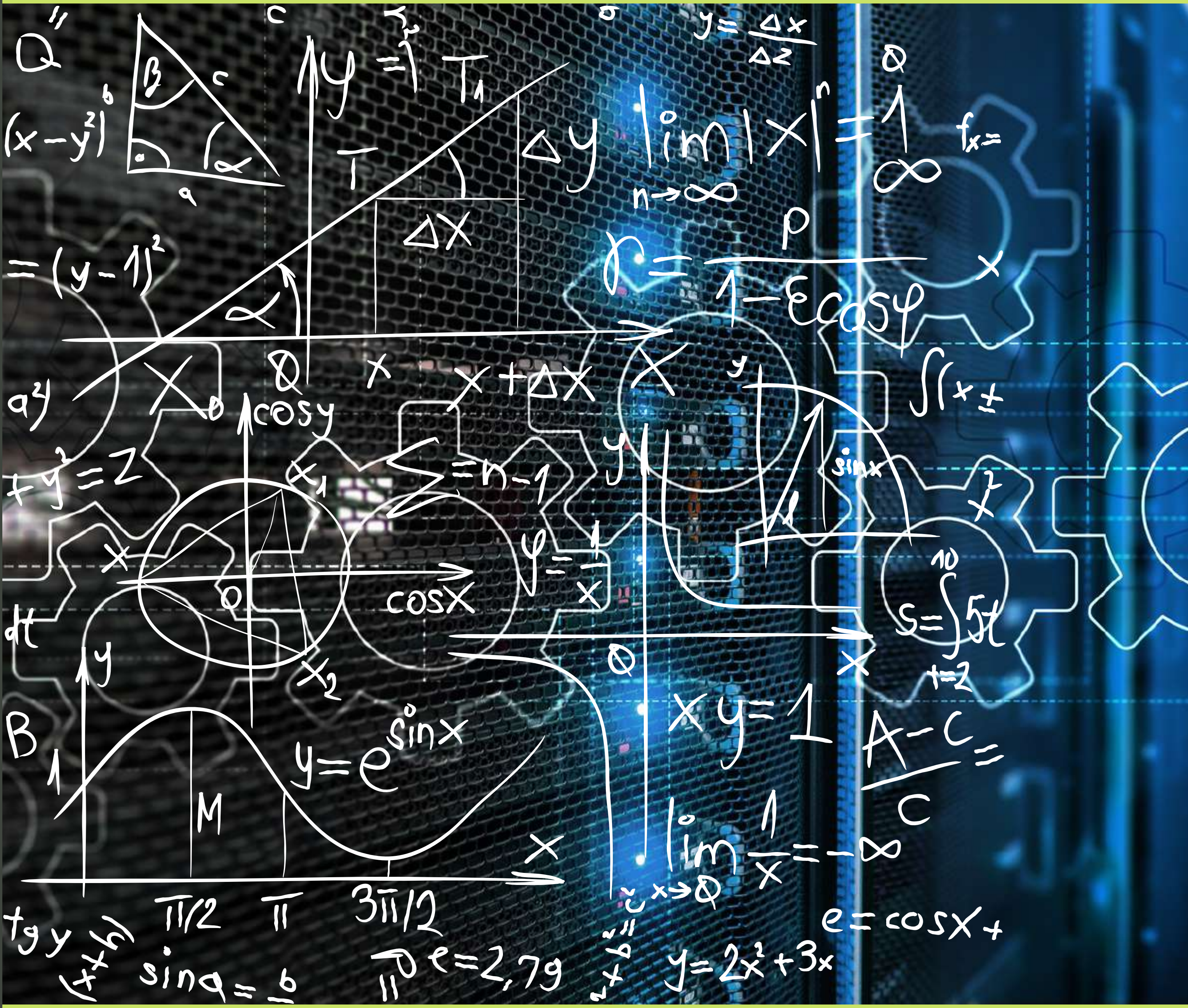
ISBN: 978-93-90753-35-2

INTEGRAL TRANSFORM AND ITS APPLICATIONS



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9 789390 753352

Published by
Novateur Publication
466, Sadashiv Peth, M.S.India-411030

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Published by Novateur Publication, Pune, M.S., India.

First Printing, 2022.

Novateur Publication
466, Sadashiv Peth, Pune, Maharashtra, India-411030
www.novateurpublication.com

BOOK ON

INTEGRAL TRANSFORM AND ITS APPLICATIONS

INDEX

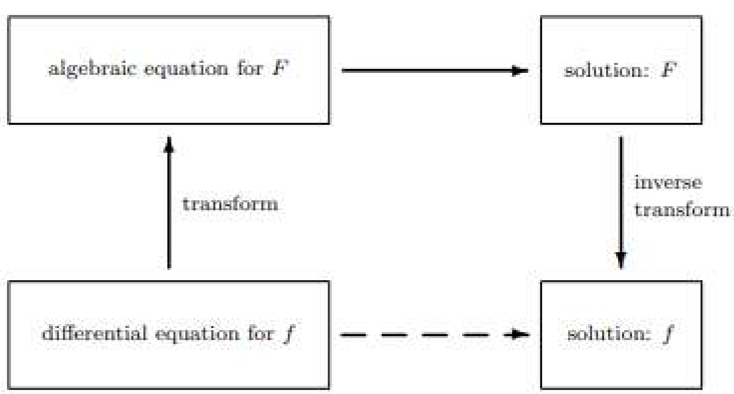
PREFACE

We present a novel complicated integral transform, the complex SEE transform, in this book. The features of this transform are studied. This complex integral transform is also used to simplify the core issue to a simple algebraic equation. The solution to this basic issue may then be determined by solving this algebraic equation and applying the inverse of this complex integral transform. Finally, the complex integral transform is employed to solve higher order ordinary differential equations. Also, we present several key engineering and physics applications. This chapter of the course teaches two incredibly effective ways to solve differential equations: the Fourier and the Laplace transforms. Beside its practical application, the Fourier transform is also of vital significance in quantum physics, establishing the relationship between the position and momentum representations of the Heisenberg commutation relations. An integral transform is valuable if it helps one to convert a difficult issue into a simpler one. The transforms we will be learning in this portion of the course are largely used to solve differential and, to a lesser degree, integral equations. The theory of Fourier series and integrals has always had considerable challenges and requires a huge mathematical apparatus in dealing with concerns of convergence. It encouraged the creation of techniques of summation, albeit they did not lead to a wholly adequate solution of the issue. ... For the Fourier transform, the inclusion of distributions (thus, the space S) is inevitable either in an explicit or hidden form. ... As a consequence one may acquire everything that is wanted from the point of view of the continuity and inversion of the Fourier transform.

CHAPTER 1

INTEGRAL TRANSFORM

This section of the course presents two incredibly strong techniques for solving differential equations: the Fourier and the Laplace transforms. The Fourier and Laplace transforms are two of the most effective methods for solving differential equations. As well since being useful in everyday life, the Fourier transform has a key role to play in quantum physics, as it is responsible for establishing correspondence between the position and momentum representations of the Heisenberg commutation relations. One of the benefits of using an integral transform is that it helps one to convert a difficult issue into a simpler one. The transforms we will be covering in this section of the course are mostly helpful for solving differential equations and, to a lesser degree, integral equations in differential equations. The concept behind a transform is really straightforward. Consider the following scenario: we are attempting to solve a differential equation involving an unknown function f. One begins by applying the transform to the differential equation in order to convert it into an equation that can be solved quickly and easily: in the case of the transform F of f, this is often an algebraic equation. This equation is then solved for F, and the inverse transform is used to determine f. This is the last step. In diagrammatic form, this circle (or square!) of concepts might be depicted as follows:



We would like to follow the dashed line, but this is often very difficult.

As a result, we choose to follow the solid line instead: while it seems to be a longer route, it has the benefit of being simple. Why else would formalization exist if not to reduce the solution of complex issues to a set of simple rules that can be followed by even the most basic of machines? We will begin by discussing Fourier series in the context of a specific example: a vibrating string, before moving on to other topics. Additionally, the separation of variables approach will be introduced in order to solve partial differential equations, which will be beneficial in the long run. We will use the Fourier integral transform to discover steady-state solutions to differential equations when the vibrating string reaches an infinite length in the limit. This will be applied specifically to the one-dimensional wave equation, which will be discussed later. We will introduce the Laplace transform to deal with transient solutions of differential equations in order to better understand them. This will subsequently be used to a variety of situations, including the solution of initial value problems, among others.

1.1 GENERAL FORM

Any transform T of the following form is considered to be an integral transform:

$$(Tf)(u) = \int_{t_1}^{t_2} f(t)\, K(t,u)\, dt$$

A function is used as the input to this transform. f, and the output is another function Tf. An integral transform is a particular kind of mathematical operator.

There are numerous useful integral transforms. Each is specified by a choice of the function K of two variables, the **kernel function**, **integral kernel** or **nucleus** of the transform.

Some kernels have an associated *inverse kernel* $K^{-1}(u,t)$ which (roughly speaking) yields an inverse transform:

$$f(t) = \int_{u_1}^{u_2} (Tf)(u)\, K^{-1}(u,t)\, du$$

A *symmetric kernel* is one that is unchanged when the two variables are permuted; it is a kernel function K such that $K(t,u) = K(u,t)$. In the theory of integral equations, symmetric kernels correspond to self-adjoint operators.

1.2 MOTIVATION FOR USE

Leaving aside the complexities of mathematical language, the idea underlying integral transformations is straightforward. There are numerous kinds of issues that are difficult to solve (or at the very least ungainly algebraically) in their original formulations, and this is true of many types of problems. An integral transform is a mathematical operation that "maps" an equation from one "domain" to another. This may be far more straightforward than manipulating and solving the problem in the original domain since the target domain is significantly smaller in size. The answer is then transferred back to the original domain using the inverse integral transform of the integral transform of the original domain.

It is possible to use integral transforms in many different applications of probability, such as the "price kernel" or stochastic discount factor, or the smoothing of data recovered through robust statistics (see kernel transform) (statistics).

Consider the Laplace transform as an example of how integral transforms may be used in a practical setting. This is a method that maps differential or integro-differential equations in the "time" domain into polynomial equations in what is referred to as the "complex frequency" domain using the concept of "complex frequency." (Complex frequency is comparable to real, physical frequency, but is more versatile in application.) Furthermore, the imaginary component I of the complex frequency s = I corresponds to the conventional concept of frequency, namely, the rate at which a sinusoid repeats, while the real component I corresponds to the degree of "damping," which is an exponential decrease in the amplitude of the sinusoid's output signal.) Complex frequency is used to cast the equation, and the problem is easily solved in the complex frequency domain (roots of polynomial equations in the complex frequency domain correspond to eigenvalues in the time domain), resulting to a "solution" that is expressed in the frequency domain. The inverse transform, which is the inverse method of the original Laplace transform, is used to get a time-domain solution, which is defined as follows: Specifically, polynomials in the complex frequency domain (usually appearing in the denominator) correlate to power series in the time domain, while axial shifts in the complex frequency domain correspond to damping by decaying exponentials in the time domain, as seen in this example.

There is widespread use of the Laplace transform in physics and, in particular, electrical engineering, where the characteristic equations that describe the behaviour of an electric circuit in the complex frequency domain correspond to linear combinations of exponentially scaled and time-shifted damped sinusoids in the time domain, as well as in other fields. Other integral transformations have a wide range of applications in a variety of scientific and mathematical areas as well.

Another usage example is the kernel in the path integral:

$$\psi(x,t) = \int_{-\infty}^{\infty} \psi(x',t')K(x,t;x',t')dx'.$$

This states that the total amplitude $\psi(x,t)$ to arrive at (x,t) is the sum (the integral) over all possible values x' of the total amplitude $\psi(x',t')$ to arrive at the point (x',t') multiplied by the amplitude to go from x' to x [i.e. $K(x,t;x',t')$]. It is often referred to as the propagator for a given system. This (physics) kernel is the kernel of the integral transform. However, for each quantum system, there is a different kernel.

Table of transforms

Table of integral transforms

Transform	Symbol	K	$f(t)$	t_1	t_2	K^{-1}	u_1	u_2
Abel transform	F, f	$\frac{2t}{\sqrt{t^2-u^2}}$		u	∞	$\frac{-1}{\pi\sqrt{u^2-t^2}}\frac{d}{du}$ [4]	t	∞
Associated Legendre transform	$\mathcal{J}_{n,m}$	$(1-x^2)^{-m/2}P_n^m(x)$		-1	1		0	∞
Fourier transform	$\mathcal{F}$	$e^{-2\pi iut}$	L_1	$-\infty$	∞	$e^{2\pi iut}$	$-\infty$	∞
Fourier sine transform	$\mathcal{F}_s$	$\sqrt{\frac{2}{\pi}}\sin(ut)$	on $[0,\infty)$, real-valued	0	∞	$\sqrt{\frac{2}{\pi}}\sin(ut)$	0	∞
Fourier cosine transform	$\mathcal{F}_c$	$\sqrt{\frac{2}{\pi}}\cos(ut)$	on $[0,\infty)$, real-valued	0	∞	$\sqrt{\frac{2}{\pi}}\cos(ut)$	0	∞
Hankel transform		$t\,J_\nu(ut)$		0	∞	$u\,J_\nu(ut)$	0	∞
Hartley transform	$\mathcal{H}$	$\frac{\cos(ut)+\sin(ut)}{\sqrt{2\pi}}$		$-\infty$	∞	$\frac{\cos(ut)+\sin(ut)}{\sqrt{2\pi}}$	$-\infty$	∞
Hermite transform	H	$e^{-x^2}H_n(x)$		$-\infty$	∞		0	∞
Hilbert transform	$\mathcal{H}il$	$\frac{1}{\pi}\frac{1}{u-t}$		$-\infty$	∞	$\frac{1}{\pi}\frac{1}{u-t}$	$-\infty$	∞
Jacobi transform	J	$(1-x)^\alpha\,(1+x)^\beta\,P_n^{\alpha,\beta}(x)$		-1	1		0	∞
Laguerre transform	L	$e^{-x}\,x^\alpha\,L_n^\alpha(x)$		0	∞		0	∞
Laplace transform	$\mathcal{L}$	e^{-ut}		0	∞	$\frac{e^{ut}}{2\pi i}$	$c-i\infty$	$c+i\infty$
Legendre transform	$\mathcal{J}$	$P_n(x)$		-1	1		0	∞
Mellin transform	$\mathcal{M}$	t^{u-1}		0	∞	$\frac{t^{-u}}{2\pi i}$ [5]	$c-i\infty$	$c+i\infty$
Two-sided Laplace transform	$\mathcal{B}$	e^{-ut}		$-\infty$	∞	$\frac{e^{ut}}{2\pi i}$	$c-i\infty$	$c+i\infty$
Poisson kernel		$\frac{1-r^2}{1-2r\cos\theta+r^2}$		0	2π			
Radon Transform	Rf			$-\infty$	∞			
Weierstrass transform	$\mathcal{W}$	$\frac{e^{-\frac{(u-t)^2}{4}}}{\sqrt{4\pi}}$		$-\infty$	∞	$\frac{e^{\frac{(u-t)^2}{4}}}{i\sqrt{4\pi}}$	$c-i\infty$	$c+i\infty$
X-ray transform	Xf			$-\infty$	∞			

In the limits of integration for the inverse transform, c is a constant whose value is dependent on the nature of the transform function being considered. The coefficient c, for example, must be bigger than the biggest real portion of the zeroes of the transform function for both one- and two-sided Laplace transformations, respectively. Note that the Fourier transform may be expressed using a variety of different notations and conventions.

1.3 BRIEF HISTORY OF INTEGRAL TRANSFORM

Integral transformations have been effectively used to a wide range of issues in applied mathematics, mathematical physics, and engineering research for almost two centuries. Integral transforms, including the Laplace and Fourier transforms, can be traced back to the celebrated work of Pierre-Simon Laplace (1749–1827) on probability theory published in the 1780s and to the monumental treatise of Joseph Fourier (1768–1830) on La Th'eorie Analytique de la Chaleur published in 1822 as the origin of the integral transforms. In reality, several of the fundamental conclusions of the Laplace transform were contained in Laplace's famous work on La Th'eorie Analytique des Probabilités, which was one of the oldest and most widely used integral transforms accessible in the mathematical literature at the time of its publication. When solving linear differential equations and integral equations, this method has shown to be quite successful. Fourier's book, on the other hand, offered the contemporary mathematical theory of heat conduction, Fourier series, and Fourier integrals, as well as applications for these concepts. Fourier's book included a startling finding, which has come to be recognised as the Fourier Integral Theorem across the world. A number of examples were provided until the conclusion was reached that an arbitrary function defined on a finite interval may be enlarged in terms of a trigonometric series, which is now commonly known as the Fourier series. When Fourier attempted to apply his new concepts to functions defined on an infinite interval, he found an integral transform and its inversion formula, which are now commonly referred to as the Fourier transform and the inverse Fourier transform, respectively.

Although Laplace and A. L. Cauchy (1789–1857) were unaware of Fourier's famed theory, they were aware of it since some of their previous work used this transformation. For his part, S. D. Poisson (1781–1840) employed the transform technique in his study on the propagation of water waves, which was done independently of others. However, it was G. W. Leibniz (1646–1716), not Newton, who was the first to propose the notion of a symbolic technique in calculus in the first place. Following Laplace's death in 1813, both J. L. Lagrange (1736–1813) and Laplace made significant contributions to symbolic approaches, which later became known as operational calculus. Despite the fact that both the Laplace and Fourier transforms were discovered in the nineteenth century, it was the British electrical engineer Oliver Heaviside (1850–1925) who made the Laplace transform widely known by using it to solve ordinary differential equations of electrical circuits and systems, and then by developing modern operational calculus as a result of his research.

The fact that the Laplace transform is simply a particular instance of the Fourier transform for a class of functions defined on the positive real axis may be significant to point out, however the Laplace transform is more straightforward than the Fourier transform for the following reasons. In the first place, the subject of convergence of the Laplace transform is significantly less delicate due to the fact that its exponentially decaying kernel exp (st) is greater than zero for all real numbers greater than zero. For the second time, the Laplace transform is an analytic function of the complex variable, and the characteristics of the Laplace transform may be investigated with a basic understanding of the theory of complex variables. Lastly, the Fourier integral formula gave us a new way to think about Laplace transforms and inverse Laplace transforms by defining them in terms of a complex contour integral that can be evaluated using the Cauchy residue theory and contour deformation in the complex plane.

It was the work of Cauchy that contained the exponential form of the Fourier Integral Theorem as

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{ik(x-y)}f(y)dydk.$$

Cauchy's work also contained the following formula for functions of the operator D:

$$\phi(D)f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(ik)e^{ik(x-y)}f(y)dydk.$$

This was a major contributor to the development of the contemporary version of the operational calculus. His well-known work, Memoire sur l'Emploi des Equations Symboliques, provides a pretty thorough exposition of symbolic approaches, which is still in print today. Many mathematicians and mathematical physicists throughout history have realised the profound relevance of the Fourier Integral Theorem, including those working in the nineteenth and twentieth century's. It is in fact widely considered as one of the most basic conclusions of contemporary mathematical analysis, and it has a broad range of applications in both the physical and engineering sciences. According to Kelvin and Tait, "...Fourier's Theorem, which is not only one of the most beautiful results of modern analysis, but may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics, is a general and important result in mathematics and physics." In order to offer a vague notion of its significance, I will merely cite sonorous vibrations, the transmission of electric signals down a telegraph wire, and the conduction of heat by the earth's crust, all of which are problems that are in their generality insoluble without it."

Oliver Heaviside (1850–1925) was the first to recognise the power and success of operational calculus in the late nineteenth century, and he was also the first to use the operational method as a powerful and effective tool for the solution of the telegraph equation and the second-order hyperbolic partial differential equations with constant coefficients, both of which he discovered in the late nineteenth century. "On Operational Techniques in Physical Mathematics," Parts I and II, which were published in The Proceedings of the Royal Society of London in 1892 and 1893, respectively, were Heaviside's first two chapters on the subject of operational methods in physical mathematics. His book on Electromagnetic Theory, published in 1899, also included information on the usage and application of operational techniques to the study of electrical circuits or networks, as well as other topics. Heaviside substituted the differential operator D d dt with the polynomial operator p and treated the latter as if it were an element of the usual rules of algebra, as seen in the diagram.

Concerns about the mathematical rigour of his operational techniques received little consideration throughout the development of his operational procedures. There was a great deal of dispute around the widespread usage of the Heaviside technique prior to its validation by the theory of the Fourier or the Laplace transform. This was analogous to the dispute that erupted in the 1920s over the widespread usage of the delta function, which was considered to be one of the most helpful mathematical techniques in Dirac's logical formulation of quantum mechanics at the time. "All electrical engineers are acquainted with the notion of a pulse," remarked P. A. M. Dirac (1902–1984), "and the -function is just a means of describing a pulse in mathematical terms." Heaviside's operator calculus in electromagnetic theory, Dirac's electrical engineering training, and his extensive knowledge of the modern theory of electrical pulses all appeared to have had a significant impact on his ingenious development of modern quantum mechanics, which was published in the journal Nature.

According to some sources, the concepts of operational techniques were derived from the classic work of Laplace, Fourier, and Cauchy, among others. Heaviside was inspired by their extraordinary efforts and built his own, less rigorous, operational mathematics in the process. Although Heaviside's calculus has achieved remarkable success as one of the most valuable mathematical approaches, contemporary mathematicians did not acknowledge Heaviside's work during his lifetime, mostly owing to a lack of mathematical rigour on the part of the authors. Oliver Heaviside's birth centenary was celebrated with a talk on Heaviside and Operational Calculus by Dr. David Heaviside. The mathematician J. L. B. Cooper (1952) disclosed some of the contentious problems surrounding Heaviside's famed work, concluding that Heaviside has "a genius for devising practical techniques of computation" as well as manipulative talent. Heaviside's theory was greatly reduced by him; according to Hertz, the four equations that are now known as Maxwell's were initially provided by Heaviside. He is considered to be one of the pioneers of vector analysis..." Cooper provided a reasonably comprehensive description of the early history of the topic, as well as a range of mathematicians' differing perspectives on Heaviside's contributions to operational calculus, while reviewing the history of Heaviside's calculus. According to Cooper, the widely reported claim that Heaviside was the one who developed operational calculus has remained a source of contention. Although there have been some debates about Heaviside's accomplishments, it is generally agreed that his most significant contribution was the development of operational calculus, which has proven to be one of the most useful mathematical devices in applied mathematics, mathematical physics, and engineering science.

From a scientific standpoint, the following remark from Lord Rayleigh seems to be the most applicable in this context: "In the mathematical research I have typically utilised such techniques as present themselves naturally to a physicist." The pure mathematician will grumble, and (it must be admitted) with some justification, about a lack of rigour in his or her work. However, there are two opposing viewpoints on this issue. For, while vital it may be to maintain a consistently high level in pure mathematics, the physicist may sometimes find it beneficial to be satisfied with arguments that are somewhat satisfying and convincing from his point of view rather than striving for perfection. It is possible that the more severe technique of the pure mathematician seems not more but less demonstrative to his mind since it is executed in a different order of thoughts.

As a result, insisting on the best possible criteria in many areas of difficulties might result in the topic being completely excluded due to the amount of space that would be necessary." With the exception of a small minority of pure mathematicians, everyone has considered Heaviside's work to be a tremendous accomplishment, despite the fact that he did not offer a formal proof of his operational calculus. In support of Heaviside, it seems that Richard P. Feynman's idea is worth quoting in its whole. In contrast to the mathematicians' focus in rigorous proof procedures, "the emphasis should be placed on how to perform the mathematics fast and easily, and what formulae are true, rather than on how to accomplish it quickly and readily." There was a certain parallel between the development of operational calculus and the development of calculus in the seventeenth century. The calculus was created by mathematicians, but they did not offer a precise formulation of it. The rigorous formulation of calculus did not appear until the nineteenth century, despite the fact that throughout the transition period, the non-rigorous presentation of calculus was still valued. It is commonly known that mathematicians of the twentieth century laid the groundwork for the Heaviside operational calculus by providing a rigorous basis. So, by any metric, Heaviside deserves a great deal of praise for his extraordinary achievements. The next phase of the development of operational calculus is defined by the endeavour to give rigorous proofs to support the justifications of heuristic approaches that have been developed.

T. J. Bromwich (1875–1930) made significant contributions to this period by being the first to effectively present the theory of complex functions in order to provide formal justification for Heaviside's calculus. His many contributions to this field include providing the formal derivation of the Heaviside expansion theorem as well as ensuring that Heaviside's operational findings were correctly interpreted. As a result of Bromwich's work, several other researchers made significant contributions to the rigorous formulation of operational calculus, including J.R. Carson, B. van der Pol, G. Doetsch, and many more. As we draw to a close our study of the historical evolution of operational calculus, we should express some scepticism about the disputed judgement of Heaviside's work.

Heaviside's operational calculus was a significant breakthrough from the perspective of applied mathematics. According to E. T. Whittaker's obituary, which serves as support for this statement, "Looking back..." we should place the operational calculus alongside Poincar's discovery of automorphic functions and Ricci's discovery of the tensor calculus as the three most important mathematical advances of the last quarter of the nineteenth century." Despite the fact that Heaviside paid little attention to concerns of mathematical rigour, he realised that operational calculus is one of the most successful and valuable mathematical tools in the field of applied mathematics and mathematics science. It was a logical progression from there to a careful mathematical investigation of integral transformations. According to this mathematical basis, procedures such as the Fourier or Laplace transforms, when applied to the data, are fundamentally similar to the present operational calculus. Besides the Mellin transform and the Hankel transform, there are many other integral transformations that are widely used to solve initial and boundary value problems involving ordinary and partial differential equations, as well as other problems in mathematics, science, and engineering. The Mellin transform is one of the most widely used integral transformations in mathematics, science, and engineering.

However, it was G. Bernhard Riemann (1826–1866) who first recognised the Mellin transform and its inversion formula in his famous memoir on prime numbers, even though Mellin (1854–1933) presented an elaborate discussion of his transform and its inversion formula in his famous memoir on prime numbers. Hermann Hankel (1839–1873), a student of G. B. Riemann, invented the Hankel transform, which uses the Bessel function as its kernel. When circular symmetry is assumed, the Hankel transform may be readily deduced from the two-dimensional Fourier transform. When dealing with boundary value issues in cylindrical polar coordinates, the Hankel transform is a logical progression. Despite the fact that the Hilbert transform was named after one of the finest mathematicians of the twentieth century, David Hilbert (1862–1943), the transform and its features were primarily researched by G. H. Hardy (1877–1947) and E. C. Titchmarsh (1899–1963) during their respective lifetimes. When studying continuous fractions, T. J. Stieltjes (1856–1894), a Dutch mathematician, developed the Stieltjes transform, which he named after himself. There are several situations in mathematics, science, and engineering where the Hilbert and Stieltjes transformations are used. The former is used in the solution of problems in fluid mechanics, signal processing, and electronics, while the latter is encountered in the solution of integral equations and moment equations.

1.4 BASIC CONCEPTS AND DEFINITIONS

The integral transform of a function $f(x)$ defined in $a \leq x \leq b$ is denoted by $I\ \{f(x)\} = F(k)$, and defined by

$$\mathscr{I}\left\{f(x)\right\} = F(k) = \int_a^b K(x,k) f(x) dx,$$

Where K(x, k), a function of two variables x and k, is termed the kernel of the transform. The operator I is commonly termed an integral transform operator or simply an integral transformation. The transform function F(k) is typically referred to as the image of the provided object function f(x), and k is termed the transform variable.

Similarly, the integral transform of a function of several variables is defined by

$$\mathscr{I}\left\{f(\mathbf{x})\right\} = F(\boldsymbol{\kappa}) = \int_S K(\mathbf{x},\boldsymbol{\kappa}) f(\mathbf{x}) d\boldsymbol{x},$$

Where x = (x1, x2,...,xn), κ = (k1, k2,...,kn), and S ⊂ Rn

Through the use of the features of Banach spaces, it is possible to create a mathematical theory of transformations of this sort. Although such a software would be of tremendous interest from a mathematical standpoint, it may not be effective in practical applications due to its limitations. The purpose of this chapter is to investigate integral transforms as operational techniques, with a particular focus on their applications. When applied to a function f(x), the integral transform operator produces another function f(x). The concept of the integral transform operator is similar to that of the well-known linear differential operator, D d dx.

(x), that is,

$$Df(x) = f'(x).$$

Usually, f

When applying the linear transformation D, (x) is referred to as the derivative or the image of f(x). There are a variety of notable integral transforms, including the Fourier, Laplace, Hankel, and Mellin transforms, which are all examples. They are defined by selecting alternative kernels K(x, k) as well as different values for the parameters a and b involved in the equation (1.4.1). Because it meets the property of linearity, it is obvious that I is a linear operator.

$$\mathscr{I}\{\alpha f(x)+\beta g(x)\} = \int_{a}^{b}\{\alpha f(x)+\beta g(x)\}K(x,k)dx$$
$$= \alpha\mathscr{I}\{f(x)\}+\beta\mathscr{I}\{g(x)\},$$

Where α and β are arbitrary constants. In order to obtain f(x) from a given F(k) = I {f(x)}, we introduce the inverse operator I −1 such that

$$\mathscr{I}^{-1}\{F(k)\} = f(x).$$

Accordingly $\mathscr{I}^{-1}\mathscr{I} = \mathscr{I}\mathscr{I}^{-1} = \mathbf{1}$ which is the identity operator It can be proved that $\mathscr{I}^{-1}$ is also a linear operator as follows

$$\begin{aligned}\mathscr{I}^{-1}\{\alpha F(k)+\beta G(k)\} &= \mathscr{I}^{-1}\{\alpha\mathscr{I}f(x)+\beta\mathscr{I}g(x)\}\\ &= \mathscr{I}^{-1}\{\mathscr{I}[\alpha f(x)+\beta g(x)]\}\\ &= \alpha f(x)+\beta g(x)\\ &= \alpha\mathscr{I}^{-1}\{F(k)\}+\beta\mathscr{I}^{-1}\{G(k)\}.\end{aligned}$$

It can also be proved that the integral transform is unique. In other words if $\mathscr{I}\{f(x)\} = \mathscr{I}\{g(x)\}$, then $f(x) = g(x)$ under suitable conditions This is known as the uniqueness theorem.

For the purpose of completing this part, we will discuss the fundamental scope and applications of integral transformation from a broad perspective. The discussion above demonstrates that an integral transformation is simply a mathematical operation in which a real or complex-valued function F is transformed into another new function F = I f or into a set of data that can be measured (or observed) experimentally, as a result of which an integral transformation is defined. As a result, the integral transform is significant because it converts a tough mathematical issue into a relatively simple problem that can be readily addressed. Initial-boundary value issues involving differential equations are studied using algebraic operations using F, which are significantly simpler than differential operators and can be solved much more quickly than differential equations.

The inverse transformation is then used to derive the answer to the original issue in the variables that were used in the original problem. In this case, the next fundamental issue is the calculation of the inverse integral transform, either precisely or approximated by the previous problem. Moreover, to make use of the integral transform technique effectively, the function f must first be reconstructed from I f = F. This is a challenging step to do in reality, as is shown in the following example. However, there are other approaches that may be used to overcome this challenge. In applications, it is often the case that the transform function F itself has some physical significance and must be examined as a separate entity. Examples include electrical engineering challenges in which the original function f (t) represents a signal that is a function of time t and is represented by the original function f (t). In signal theory, the Fourier transform f (t) of a signal f (t) is used to describe the frequency spectrum of the signal f (t), and it is physically useful as a temporal representation of the signal. In fact, it is often more vital to work with f than it is to work with f. The inverse Fourier transform, on the other hand, may be used to reconstruct the original signal f (t) given the frequency spectrum, f (t).

CHAPTER 2

FOURIER TRANSFORMS AND THEIR APPLICATIONS

"The profound study of nature is the most fertile source of mathematical discoveries."

"The theory of Fourier series and integrals has always had major difficulties and necessitated a large mathematical apparatus in dealing with questions of convergence. It engendered the development of methods of summation, although these did not lead to a completely satisfactory solution of the problem. ... For the Fourier transform, the introduction of distributions (hence, the space S) is inevitable either in an explicit or hidden form. ... As a result one may obtain all that is desired from the point of view of the continuity and inversion of the Fourier transform."

2.1 INTRODUCTION

When the Fourier transform, the Fourier cosine transform, or the Fourier sine transform are used in conjunction with other linear boundary value and initial value problems in applied mathematics, mathematical physics, and engineering science, many linear boundary value and initial value problems can be effectively solved. The following are some of the reasons why these transformations are very beneficial for solving differential or integral problems. The first step is to replace these equations with simple algebraic equations that will allow us to discover the solution of the transform function.

After that, by inverting the transform solution, the solution of the supplied equation is found in the original variables of the equation. Second, the Fourier transform of the elementary source term is utilized to determine the fundamental solution, which serves as an illustration of the fundamental concepts underlying the development and execution of Green's functions and their applications. In addition, when the transform solution is paired with the convolution theorem, it is possible to get an elegant formulation of the solution for both the boundary value and the starting value issues. The formal derivation of the Fourier integral for mulas serves as the starting point for this chapter. Using these principles, the Fourier, Fourier cosine, and Fourier sine transforms are defined and employed in many applications. This is followed by a full examination of the fundamental operational features of these transforms, which includes several examples and illustrations. Convolution and its fundamental features are discussed in detail. Sections 2.10 and 2.11 are concerned with the application of the Fourier transform to the solution of ordinary differential equations and integral equations, respectively. Section 2.10: In Section 2.12, the Fourier transform technique is used to solve a broad range of partial differential equations, and the results are presented. With very minor modifications, the approach presented in this and subsequent parts may be used with little or no change to a variety of various types of starting and boundary value issues that are found in applications. After introducing the Fourier cosine and sine transforms in Section 2.13, Sections 2.14 and 2.15 go into further detail on the characteristics and applications of these transforms, respectively. Fourier transforms are used to evaluate definite integrals, which is followed by the assessment of definite integrals. Section 2.17 is dedicated to the use of Fourier transformations in the field of mathematical statistics and probability. Section 2.18 discusses the numerous Fourier transforms and the applications of these transformations.

2.2 THE FOURIER INTEGRAL FORMULAS

A function f(x) is said to satisfy Dirichlet's conditions in the interval $-a < x < a$, if

(i) F(x) has only a finite number of finite discontinuities in $-a<x< a$ and has no infinite discontinuities.

(ii) F(x) has only a finite number of maxima and minima in $-a<x<a$.

From the theory of Fourier series we know that if f(x) satisfies the Dirichlet conditions in a<x<a, it can be represented as the complex Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \exp(in\pi x/a), \qquad (2.2.1)$$

Where the coefficients are

$$a_n = \frac{1}{2a}\int_{-a}^{a} f(\xi)\exp(-in\pi\xi/a)d\xi. \qquad (2.2.2)$$

This representation is evidently periodic of period 2a in the interval. However, the right-hand side of (2.2.1) cannot represent f(x) outside the interval $-a < x<a$ unless f(x) is periodic of period 2a.

Thus problems on finite intervals lead to Fourier series, and problems on the whole line $-\infty < x < \infty$ lead to the Fourier integrals. We now attempt to find an integral representation of a nonperiodic function f(x) in $(-\infty, \infty)$ by letting $a \to \infty$. As the interval grows ($a \to \infty$) the values kn = nπ a become closer together and form a dense set. If we write δkn = (kn+1 − kn) = π a and substitute coefficients an into (2.2.1), we obtain

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (\delta k_n) \left[\int_{-a}^{a} f(\xi) \exp(-i\xi k_n) d\xi \right] \exp(ixk_n). \qquad (2.2.3)$$

In the limit as $a \to \infty$, kn becomes a continuous variable k and δkn becomes dk. Consequently, the sum can be replaced by the integral in the limit and (2.2.3) reduces to the result

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\xi) e^{-ik\xi} d\xi \right] e^{ikx} dk. \qquad (2.2.4)$$

This is referred to as the Fourier integral formula, which is well-known. (2.2.4) is accurate and valid for functions that are piecewise continuously differentiable in every finite interval and are absolutely integrable on the whole real line, despite the fact that the reasons presented above do not provide an exhaustive proof. When a function f(x) is absolutely integrable on (x, y), it is said to be absolutely integrable on (x, y).

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty \qquad (2.2.5)$$

Exists It can be shown that the formula (2.2.4) is valid under more general conditions. The result is contained in the following theorem:

THEOREM 2.2.1 (The Fourier Integral Theorem)

If f(x) satisfies Dirichlet’s conditions in $(-\infty, \infty)$, and is absolutely integrable on $(-\infty, \infty)$, then the Fourier integral (2.2.4) converges to the function 1 2 [f(x + 0) + f(x − 0)] at a finite discontinuity at x. In other words,

$$\frac{1}{2}[f(x+0)+f(x-0)]=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\left[\int_{-\infty}^{\infty} f(\xi)e^{-ik\xi}d\xi\right] dk. \qquad (2.2.6)$$

This is usually called the Fourier integral theorem.

If the function f(x) is continuous at point x, then f(x + 0) = f(x − 0) = f(x), then (2.2.6) reduces to (2.2.4).

The Fourier integral theorem was first presented in Fourier's classic book entitled La The'orie Analytique da la Chaleur (1822), and its profound relevance was acknowledged by both mathematicians and mathematical physicists at the time of its publication. It is true that this theorem is one of the most monumental conclusions of contemporary mathematical analysis, and that it has several physical and practical applications as well. Using trigonometric functions, we may describe the exponential component exp[ik(x)] in (2.2.4) in terms of even and odd natures of the cosine and sine functions, respectively, as functions of k. As a result, (2.2.4) can be expressed as

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty} dk \int_{-\infty}^{\infty} f(\xi)\cos k(x-\xi)d\xi. \qquad (2.2.7)$$

There are many variations on the Fourier integral formula. This is one of them. f(x) disappears relatively quickly as |x| in many physical situations, ensuring the presence of the repeated integrals as they are written in the previous sentence. We will now assume that f(x) is an even function and will extend the cosine function in (2.2.7) to achieve the following result:

$$f(x)=f(-x)=\frac{2}{\pi}\int_{0}^{\infty}\cos kx\, dk \int_{0}^{\infty} f(\xi)\cos k\xi\, d\xi. \qquad (2.2.8)$$

This is called the Fourier cosine integral formula. Similarly, for an odd function f(x), we obtain the Fourier sine integral formula

$$f(x)=-f(-x)=\frac{2}{\pi}\int_{0}^{\infty}\sin\, kx\, dk \int_{0}^{\infty} f(\xi)\sin k\xi\, d\xi. \qquad (2.2.9)$$

These integral formulas were discovered independently by Cauchy in his work on the propagation of waves on the surface of water.

2.3 DEFINITION OF THE FOURIER INTEGRAL AND EXAMPLES

We use the Fourier integral formula (2.2.4) to give a formal definition of the Fourier transform.

DEFINITION 2.3.1 The Fourier transform of f(x) is denoted by F{f(x)} = F(k), k ∈ R, and defined by the integral

$$\mathscr{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx, \qquad (2.3.1)$$

In this case, F is referred to as the Fourier transform operator or the Fourier transformation, and the factor 1 2 is derived by dividing the factor 1 2 involved in the Fourier transformation (2.2.4). This is referred to as the complex Fourier transform in certain circles. When f(x) is absolutely integrable on (x, y), it is sufficient for f(x) to be represented by a Fourier transform. The convergence of the integral (2.3.1) arises immediately from the fact that f(x) is absolutely integrable, as shown in the previous section. In actuality, the integral converges evenly with respect to k when k is increased. Physically, the Fourier transform F(k) may be thought of as a superposition of an unlimited number of sinusoidal oscillations with distinct wavenumbers k (or different wavelengths k = 2k) that are integrated together.

This means that only completely integrable functions are allowed to be included in the formulation of the Fourier transform. A lot of physical applications will be unable to operate under this constraint. However, despite the fact that many basic and popular functions such as the constant function, the trigonometric functions sin and cos axe, the exponential function, and the constant function are regularly encountered in applications, they do not have Fourier transforms. When f(x) is one of the basic functions listed above, the integral in (2.3.1) does not converge properly. This is a highly problematic characteristic in the theory of Fourier transforms, and it ought to be addressed. This unpleasant aspect, on the other hand, may be overcome by using a natural extension of the definition of the Fourier transform of a generalised function, f(x), in the context of a generalised function (2.3.1). Following in the footsteps of Lighthill (1958) and Jones (1982), we will briefly review the theory of the Fourier transforms of good functions in this section. The inverse Fourier transform, indicated by the symbol F1F(k) = f(x), is described by the following equation:

$$\mathscr{F}^{-1}\{F(k)\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} F(k)\, dk, \qquad (2.3.2)$$

In the case of F1, which is referred to as the inverse Fourier transform operator, it should be noted that both F and F1 are linear integral operators. In applied mathematics, the variable x is often represented by a space variable, and the variable k(= 2) is typically represented by a wavenumber variable, where k is the wavelength. In electrical engineering, on the other hand, x is replaced by the time variable t, and k is replaced by the frequency variable (= 2v), where v is the frequency in cycles per second and t is the time variable. The function F(t) = Ff(t) is referred to as the spectrum of the time signal function f (time signal function f) (t). The Fourier transform pairs are defined somewhat differently in electrical engineering literature than they are in mathematics.

$$\mathscr{F}\{f(t)\} = F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-2\pi\nu i t} dt, \qquad (2.3.3)$$

And

$$\mathscr{F}^{-1}\{F(\nu)\} = f(t) = \int_{-\infty}^{\infty} F(\nu) e^{2\pi i \nu t} d\nu = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega, \qquad (2.3.4)$$

The angular frequency is defined as v = 2v where v = 2v is the value of v. As a result of the Fourier integral formula, every function of time f(t) that has a Fourier transform may be defined equally well by its spectrum, according to the formula. When viewed in terms of its physical representation, the signal f(t) may be thought of as an integral superposition of an unlimited number of sinusoidal oscillations with varying frequencies and complex amplitudes 1 2 F(t) (see Figure 1). The spectral resolution of the signal f(t) is denoted by the equation (2.3.4), while the spectral density is denoted by the equation (2.3.4). The Fourier transform is a mathematical transformation that transfers a function (or signal) of time t to a function of frequency f. It works in the same way that a periodic function decomposes into harmonic components when using the

Fourier series expansion; similarly, the Fourier transform generates a function (or signal) of a continuous variable whose value corresponds to the frequency content of the original signal when using the Fourier transform. As a result, the Fourier transform has been successfully used to the analysis of the shape of time-varying signals in electrical engineering and seismology, among other fields.

Next we give examples of Fourier transforms.

Example 2.3.1 Find the Fourier transform of exp(−ax2). Then find $\int_{-\infty}^{\infty} x^2 e^{-ax^2}\,dx.$

In fact, we prove

$$F(k) = \mathscr{F}\{\exp(-ax^2)\} = \frac{1}{\sqrt{2a}} \exp\left(-\frac{k^2}{4a}\right), \quad a > 0. \qquad (2.3.5)$$

Here we have, by definition,

$$\begin{aligned} F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx-ax^2}\,dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-a\left(x+\frac{ik}{2a}\right)^2 - \frac{k^2}{4a}\right] dx \\ &= \frac{1}{\sqrt{2\pi}} \exp(-k^2/4a) \int_{-\infty}^{\infty} e^{-ay^2}\,dy = \frac{1}{\sqrt{2a}} \exp\left(-\frac{k^2}{4a}\right), \end{aligned}$$

In this case, the modification of the variable y = x + ik 2a is provided as an example. Despite the fact that the preceding conclusion is true, the change in variable may be justified using the complex analysis approach since (ik/2a) is a complex number. If a = 1 and 2 then

$$\mathscr{F}\{e^{-x^2/2}\} = e^{-k^2/2}. \qquad (2.3.6)$$

This shows F{f(x)} = f(k). Such a function is said to be self-reciprocal under the Fourier transformation. Graphs of f(x) = exp(−ax2) and its Fourier transform is shown in Figure 2.1 for a = 1.

Alternatively, (2.3.5) can be proved as follows:

$$F'(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-ix) e^{-ikx-ax^2} dx$$
$$= \frac{i}{2a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[(-2ax) e^{-ax^2} \right] e^{-ikx} dx$$

Which is, integrating by parts

$$F'(k) = \frac{i}{2a\sqrt{2\pi}} \left\{ \left[e^{-ax^2-ikx} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (ik) e^{-ax^2-ikx} \, dx \right\}$$
$$= -\frac{k}{2a} F(k).$$

The solution for F(k) is F(k) = A(k)e− k2 4a so that

$$A(0) = F(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} dx = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{a}} = \frac{1}{\sqrt{2a}}.$$

Thus, F(k) $= \frac{1}{\sqrt{2a}} e^{-\frac{k^2}{4a}}$

Using (2.3.5), we prove that

$$I = \int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \quad a > 0.$$

It follows from (2.3.5) that

$$\int_{-\infty}^{\infty} e^{-ikx-ax^2} dx = \sqrt{2\pi} F(k) = \sqrt{\frac{\pi}{a}} e^{-\frac{k^2}{4a}}.$$

This is true for all k, and hence, putting k = 0 we obtain the desired result. Differentiating once under the integral sign with respect to a gives

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}} = \frac{1}{2a}\sqrt{\frac{\pi}{a}}.$$

Differentiating the integral I, n times with respect to a, yields

$$\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = \frac{1.3.5......(2n-1)}{2^n}\sqrt{\frac{\pi}{a^{2n+1}}}$$
$$= \frac{1.3.5......(2n-1)}{(2a)^n}\sqrt{\frac{\pi}{a}}.$$

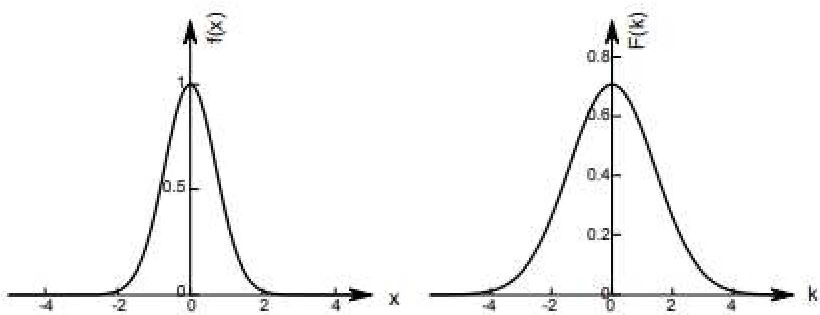


Figure 2.1 Graphs of f(x) = exp(−ax2) and F(k) with a = 1.

Example 2.3.2 Find the Fourier transform of exp(−a|x|), i.e.,

$$\mathscr{F}\{\exp(-a|x|)\} = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{(a^2+k^2)}, \quad a > 0. \qquad (2.3.7)$$

Here we can write

$$\mathscr{F}\left\{e^{-a|x|}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|-ikx}dx$$
$$= \frac{1}{\sqrt{2\pi}}\left[\int_{0}^{\infty} e^{-(a+ik)x}dx + \int_{-\infty}^{0} e^{(a-ik)x}dx\right]$$
$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+ik}+\frac{1}{a-ik}\right] = \sqrt{\frac{2}{\pi}}\frac{a}{(a^2+k^2)}.$$

We note that f(x) = exp(−a|x|) decreases rapidly at infinity, it is not differentiable at x = 0. Graphs of f(x) = exp(−a|x|) and its Fourier transform is displayed in Figure 2.2 for a = 1.

Example 2.3.3 Find the Fourier transform of

$$f(x) = \left(1-\frac{|x|}{a}\right)H\left(1-\frac{|x|}{a}\right),$$

Where H(x) is the Heaviside unit step function defined by

$$H(x) = \left\{\begin{matrix} 1, & x>0 \\ 0, & x<0 \end{matrix}\right\}. \qquad (2.3.8)$$

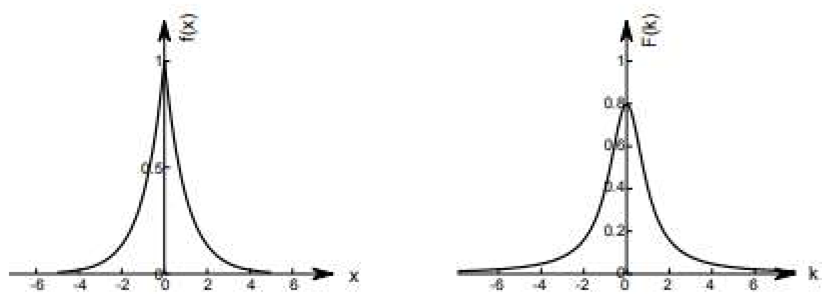


Figure 2.2 Graphs of f(x) = exp(−a|x|) and F(k) with a = 1.

Or, more generally,

$$H(x-a) = \left\{\begin{matrix} 1, & x>a \\ 0, & x<a \end{matrix}\right\}, \qquad (2.3.9)$$

Where a is a fixed real number. So the Heaviside function H(x − a) has a finite discontinuity at x = a.

$$\mathscr{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-ikx}\left(1-\frac{|x|}{a}\right)dx = \frac{2}{\sqrt{2\pi}}\int_{0}^{a}\left(1-\frac{x}{a}\right)\cos kx\,dx$$
$$= \frac{2a}{\sqrt{2\pi}}\int_{0}^{1}(1-x)\cos(akx)dx = \frac{2a}{\sqrt{2\pi}}\int_{0}^{1}(1-x)\frac{d}{dx}\left(\frac{\sin akx}{ak}\right)dx$$
$$= \frac{2a}{\sqrt{2\pi}}\int_{0}^{1}\frac{\sin(akx)}{ak}dx = \frac{a}{\sqrt{2\pi}}\int_{0}^{1}\frac{d}{dx}\left[\frac{\sin^2\left(\frac{akx}{2}\right)}{\left(\frac{ak}{2}\right)^2}\right]dx$$
$$= \frac{a}{\sqrt{2\pi}}\frac{\sin^2\left(\frac{ak}{2}\right)}{\left(\frac{ak}{2}\right)^2}. \qquad (2.3.10)$$

Example 2.3.4 Find the Fourier transform of the characteristic function $\chi_{[-a,a]}(x)$, where

$$\chi_{[-a,a]}(x) = H(a-|x|) = \left\{\begin{matrix} 1, & |x|<a \\ 0, & |x|>a \end{matrix}\right\}. \qquad (2.3.11)$$

We have

$$F_a(k) = \mathscr{F}\{\chi_{[-a,a]}(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\chi_{[-a,a]}(x)\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-ikx}\,dx = \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right). \quad (2.3.12)$$

Graphs of $f(x) = \chi_{[-a,a]}(x)$ and its Fourier transform are shown in Figure 2.3 for a = 1.

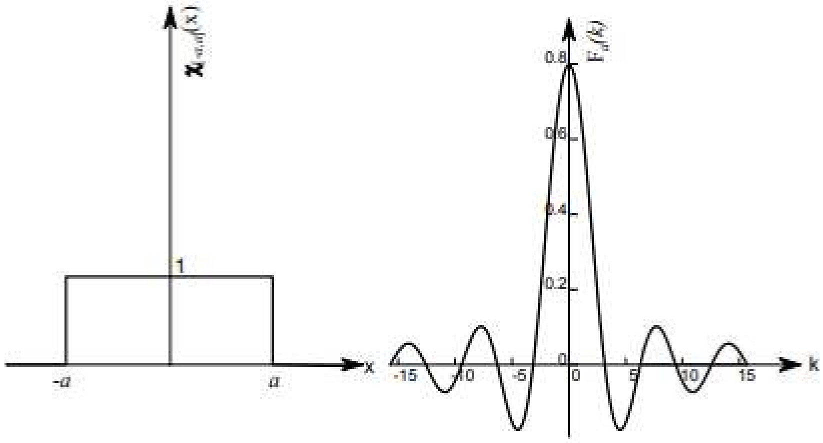


Figure 2.3 Graphs of χ[−a,a](x) and Fa(k) with a = 1.

2.4 FOURIER TRANSFORMS OF GENERALIZED FUNCTIONS

f(x) in (2.3.1) may be treated as a generalised function, which is the most logical method to describe the Fourier transform of a generalised functional form. As a result, every generalised function has a Fourier transform and an inverse Fourier transform, and the ordinary functions whose Fourier transforms are of relevance constitute a subset of the generalised functions, which is an advantage. The introduction to the topic of generalised functions is provided by the well-known works by Lighthill (1958) and Jones (1982), which are not discussed in great length here. G(x) is an excellent function in C(R) if and only if g(x) and all of its derivatives decay to zero faster than |x| N as |x| for all N > 0. A good function is one that decays sufficiently quickly that g(x) and all of its derivatives decay to zero faster than |x| N as |x| for all N > 0.

DEFINITION 2.4.1 Suppose a real or complex valued function g(x) is defined for all x ∈ R and is infinitely differentiable everywhere, and suppose that each derivative tends to zero as |x|→∞ faster that any positive power of % x−1& , or in other words, suppose that for each positive integer N and n,

$$\lim_{|x|\to\infty} x^N g^{(n)}(x) = 0,$$

Then g(x) is called a good function.

S is often used to indicate the class of excellent functions in a mathematical context. It is crucial that the good functions are used in Fourier analysis since they allow for simplified variants of the inversion, convolution and differentiation theorems as well as many others to be proved without encountering the issue of convergence. As a result of the quick decay and infinite differentiability qualities of good functions, it is also possible to have a good function that is the Fourier transform of another good function. Good functions are also significant in the theory of generalised functions, where they play a vital role. A good function with limited support is a sort of good function that is distinct from other types of good functions and has an essential role to play in the theory of generalised functions. In addition, good functions possess the following critical characteristics. A good function is formed by taking the sum (or difference) of two excellent functions. Two excellent functions are combined to form a product and a convolution of two good functions. Whenever a good function exists, its derivative is also a good function; hence, for any non-negative integers n, g(x) is a good function for which g(x) is a good function. For any p in 1 p p, a good function belongs to Lp (a class of pth power Lebesgue integrable functions) and is thus a good function. The integral of a good function does not imply that the function is excellent. On the other hand, if (x) is a good function, then the function g defined for every x by (x) is also a good function.

$$g(x) = \int_{-\infty}^{x} \phi(t)\, dt$$

is a good function if and only if $\int_{-\infty}^{\infty} \phi(t)$dt exists.

Good functions are not only continuous, but are also uniformly continuous in R and absolutely continuous in R. However, a good function cannot be necessarily represented by a Taylor series expansion in every interval. As an example, consider a good function of bounded support

$$g(x) = \left\{ \begin{array}{ll} \exp[-(1-x^2)^{-1}], & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1 \end{array} \right\}.$$

The function g is infinitely differentiable at x = ±1, as it must be in order to be good. It does not have a Taylor series expansion in every interval, because a Taylor expansion based on the various derivatives of g for any point having |x| > 1 would lead to zero value for all x.

For example, exp(−x2), x exp(−x2), % 1 + x2&−1 exp(−x2), and sech2x are good functions, while exp(−|x|) is not differentiable at x = 0, and the function % 1 + x2&−1 is not a good function as it decays too slowly as |x|→∞. A sequence of good functions, {fn(x)} is called regular if, for any good function g(x),

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx \qquad (2.4.1)$$

Exists For example, fn(x) = 1 n φ(x) is a regular sequence for any good function φ(x), if

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \lim_{n\to\infty}\frac{1}{n}\int_{-\infty}^{\infty} \phi(x)\, g(x)\, dx = 0\,.$$

Two regular sequences of good functions are identical if and only if the limit (2.4.1) exists for each good function g(x) and is the same for both sequences of good functions. In mathematics, a generalised function, denoted by f(x), is a regular sequence of good functions, and two generalised functions are equal if their defining sequences are same. As a result, generalised functions are only defined in terms of their effect on the integrals of good functions if and only if

$$\langle f,\, g\rangle = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx = \lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \lim_{n\to\infty}\langle f_n,\, g\rangle \quad (2.4.2)$$

when you have any excellent function, such as g(x), and you use the symbol f, g, you are denoting the interaction between f(x) and the good function g(x), or the symbol f, g is used to symbolise the number that is associated with f when you have any good function. For example, if f(x) is an ordinary function with the property that percent 1+x2&N f(x) is integrable in (,) for some N, the generalised function f(x) equivalent to the ordinary function is defined as any sequence of good functions fn(x) such that, for any good function g, percent 1 + x2&N f(x) is integrable in (x),

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx \qquad (2.4.3)$$

For example, the generalized function equivalent to zero can be represented by either of the sequences

$\left\{\frac{\phi(x)}{n}\right\}$ and $\left\{\frac{\phi(x)}{n^2}\right\}$.

The unit function, I(x), is defined by

$$\int_{-\infty}^{\infty} I(x)\, g(x)\, dx = \int_{-\infty}^{\infty} g(x)\, dx \qquad (2.4.4)$$

For any good function g(x). A very important and useful good function that defines the unit function is $\left\{\exp\left(-\frac{x^2}{4n}\right)\right\}$. Thus, the unit function is the generalized function that is equivalent to the ordinary function f(x) = 1.

The Heaviside function, H(x), is defined by

$$\int_{-\infty}^{\infty} H(x)\, g(x)\, dx = \int_{0}^{\infty} g(x)\, dx. \qquad (2.4.5)$$

The generalized function H(x) is equivalent to the ordinary unit function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases} \qquad (2.4.6)$$

It is not necessary to consider the value of H(x) at x = 0 in this context since generalised functions are defined by the action on integrals of good functions in this context. The sign function, denoted by the symbol sgn(x), is defined as

$$\int_{-\infty}^{\infty} \mathrm{sgn(x)\, g(x)\, dx} = \int_{0}^{\infty} g(x)\, dx - \int_{-\infty}^{0} g(x)\, dx \qquad (2.4.7)$$

For any good function g(x). Thus, sgn(x) can be identified with the ordinary function

$$\mathrm{sgn(x)} = \begin{cases} -1, & x < 0, \\ +1, & x > 0. \end{cases} \qquad (2.4.8)$$

In fact, sgn(x) = 2 H(x) − I(x) can be seen as follows:

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{sgn(x)\,g(x)\,dx} &= \int_{-\infty}^{\infty} [2\,H(x) - I(x)]\; g(x)\,dx \\ &= 2\int_{-\infty}^{\infty} H(x)\,g(x)\,dx - \int_{-\infty}^{\infty} I(x)\,g(x)\,dx \\ &= 2\int_{0}^{\infty} g(x)\,dx - \int_{-\infty}^{\infty} g(x)\,dx \\ &= \int_{0}^{\infty} g(x)\,dx - \int_{-\infty}^{0} g(x)\,dx\end{aligned}$$

In 1926, Dirac introduced the delta function, δ(x), having the following properties

$$\begin{aligned}&\delta(x) = 0, \qquad x \neq 0, \\ &\int_{-\infty}^{\infty} \delta(x)dx = 1.\end{aligned} \qquad (2.4.9)$$

The Dirac delta function, δ(x) is defined so that for any good function φ(x),

$$\int_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx = \phi(0).$$

There is no regular function that can be used to replace the delta function in this case. There are no regular functions in classical mathematics that can satisfy the requirements of property (2.4.9). The delta function is not a function in the classical sense, such as an ordinary function f(x), and (x) is not the value of at the given point in time. The function may, however, be viewed as a function in the generalised sense, and in fact, it is referred to as a generalised function or a generalised distribution. In current mathematics, the notion of the delta function is straightforward and straightforward. It is very valuable in the fields of physics and engineering. Physically, the delta function represents a point mass, which is a particle of unit mass that is positioned at the origin of the function. In this context, it is referred to as a mass-density function (or a mass-density function). This leads to the conclusion for a point particle, which may be thought of as the limit of a succession of continuous distributions that grow more and more concentrated as the particle gets closer to the limit. The fact that x is not a function in the classical sense does not preclude the use of a series of conventional functions to approximate it. Consider the following series of functions as an illustration:

$$\delta_n(x) = \sqrt{\frac{n}{\pi}} \exp(-nx^2), \quad n = 1, 2, 3, \ldots \qquad (2.4.10)$$

Clearly, δn(x) → 0 as n → ∞ for any x = 0 and δn(0) → ∞ as n → ∞ as shown in Figure 2.4. Also, for all n = 1, 2, 3,...,

$$\int_{-\infty}^{\infty} \delta_n(x) dx = 1$$

And

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \delta_n(x) dx = \int_{-\infty}^{\infty} \delta(x) dx = 1$$

As was to be anticipated As a result, the delta function may be thought of as the limit of a series of regular functions, and we can express it as follows:

$$\delta(x) = \lim_{n \to \infty} \sqrt{\frac{n}{\pi}} \exp(-nx^2). \qquad (2.4.11)$$

Sometimes, the delta function δ(x) is defined by its fundamental property

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\, dx = f(a), \qquad (2.4.12)$$

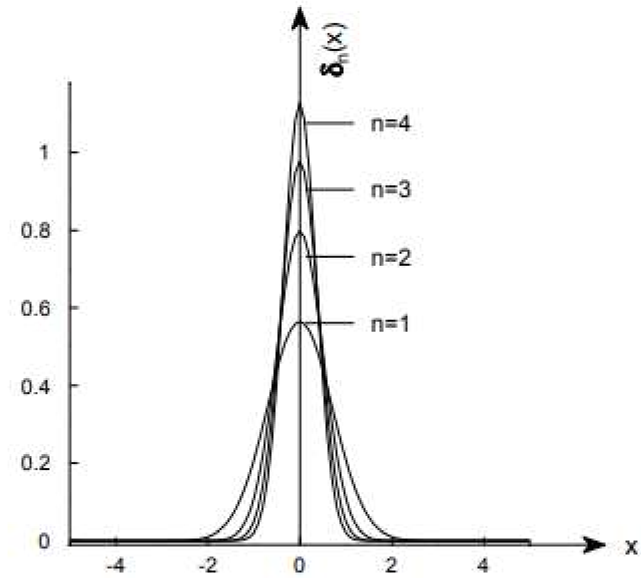


Figure 2.4 the sequence of delta functions, δn(x).

Where f(x) is continuous in any interval containing the point x = a. Clearly,

$$\int_{-\infty}^{\infty} f(a)\delta(x-a)\,dx = f(a)\int_{-\infty}^{\infty}\delta(x-a)\,dx = f(a). \qquad (2.4.13)$$

Thus, (2.4.12) and (2.4.13) lead to the result

$$f(x)\delta(x-a) = f(a)\delta(x-a). \qquad (2.4.14)$$

The following results are also true

$$x\,\delta(x) = 0 \qquad (2.4.15)$$

$$\delta(x-a) = \delta(a-x). \qquad (2.4.16)$$

Result (2.4.16) shows that δ(x) is an even function. Clearly, the result

$$\int_{-\infty}^{x}\delta(y)\,dy = \begin{Bmatrix} 1, & x>0 \\ 0, & x<0 \end{Bmatrix} = H(x)$$

Shows that

$$\frac{d}{dx}H(x) = \delta(x). \qquad (2.4.17)$$

The Fourier transform of the Dirac delta function is

$$\mathscr{F}\{\delta(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\delta(x)\,dx = \frac{1}{\sqrt{2\pi}}. \qquad (2.4.18)$$

Hence

$$\delta(x) = \mathscr{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}}\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk. \qquad (2.4.19)$$

This is an integral form of the delta function, which is a mathematical function that is widely utilised in quantum mechanics. As an alternative, the number (2.4.19) may be expressed as

$$\delta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dx. \qquad (2.4.20)$$

The Dirac delta function, δ(x) is defined so that for any good function g(x),

$$\langle \delta, g\rangle = \int_{-\infty}^{\infty} \delta(x)\,g(x)\,dx = g(0). \qquad (2.4.21)$$

Derivatives of generalized functions are defined by the derivatives of any equivalent sequences of good functions. We can integrate by parts using any member of the sequences and assuming g(x) vanishes at infinity. We can obtain this definition as follows:

$$\begin{aligned}\langle f', g\rangle &= \int_{-\infty}^{\infty} f'(x)\,g(x)\,dx \\ &= [f(x)\,g(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\,g'(x)\,dx = -\langle f, g'\rangle\,.\end{aligned}$$

The derivative of a generalized function f is the generalized function f defined by

$$\langle f',\, g\rangle = -\,\langle f,\, g'\rangle \qquad (2.4.22)$$

g is a good function for every good function The differential calculus of generalised functions may be created quickly and readily using locally integrable functions as the starting point. A generalised function (or distribution) defined by corresponds to every locally integrable function (or distribution) defined by

$$\langle f,\, \phi\rangle = \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx \qquad (2.4.23)$$

Where φ is a test function in R → C with bounded support (φ is infinitely differentiable with its derivatives of all orders exist and are continuous).

The derivative of a generalized function f is the generalized function fdefined by

$$\langle f',\, \phi\rangle = -\,\langle f,\, \phi'\rangle \qquad (2.4.24)$$

for all test functions φ. This definition follows from the fact that

$$\begin{aligned}\langle f',\, \phi\rangle &= \int_{-\infty}^{\infty} f'(x)\,\phi(x)\,dx \\ &= [f(x)\,\phi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\,\phi'(x)\,dx = -\,\langle f,\, \phi'\rangle\end{aligned}$$

This result was derived by the process of integration by parts and the fact that disappears at infinity. It is simple to verify that H is correct.

(x) = δ(x), for

$$\begin{aligned}\langle H',\, \phi\rangle &= \int_{-\infty}^{\infty} H'(x)\,\phi(x)\,dx = -\int_{-\infty}^{\infty} H(x)\,\phi'(x)\,dx \\ &= -\int_{0}^{\infty} \phi'(x)\,dx = -\,[\phi(x)]_{0}^{\infty} = \phi(0) = \langle \delta,\, \phi\rangle\,.\end{aligned}$$

Another result is

$$\langle \delta', \phi \rangle = -\int_{-\infty}^{\infty} \delta(x)\, \phi'(x)\, dx = -\phi'(0)\,.$$

It is easy to verify

$$f(x)\,\delta(x) = f(0)\,\delta(x)\,.$$

We next define |x| = x sgn(x) and calculate its derivative as follows. We have

$$\begin{aligned}\frac{d}{dx}|x| = \frac{d}{dx}\{x\,\mathrm{sgn(x)}\} &= x\,\frac{d}{dx}\{\mathrm{sgn(x)}\} + \mathrm{sgn(x)}\,\frac{\mathrm{dx}}{\mathrm{dx}}\\ &= x\,\frac{d}{dx}\{2\,H(x) - I(x)\} + \mathrm{sgn(x)}\\ &= 2x\,\delta(x) + \mathrm{sgn(x)} = \mathrm{sgn(x)} \qquad (2.4.25)\end{aligned}$$

Which is, by sgn(x) = 2 H(x) − I(x) and x δ(x) = 0.

Similarly, we can show that

$$\frac{d}{dx}\{\mathrm{sgn(x)}\} = 2H'(x) = 2\delta(x). \qquad (2.4.26)$$

Assuming that we can demonstrate that (2.3.1) holds for good functions, we may conclude that it holds for generalised functions as well.

THEOREM 2.4.1 The Fourier transform of a good function is a good function.

PROOF The Fourier transform of a good function f(x) exists and is given by

$$\mathscr{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, f(x)\, dx. \qquad (2.4.27)$$

Differentiating F(k) n times and integrating N times by parts, we get

$$\left|F^{(n)}(k)\right| \le \left|\frac{(-1)^N}{(-ik)^N}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,\frac{d^N}{dx^N}\left\{(-ix)^n f(x)\right\}dx\right|$$
$$\le \frac{1}{|k|^N}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|\frac{d^N}{dx^N}\left\{x^n f(x)\right\}\right|dx.$$

Evidently, all derivatives tend to zero as fast as |k| −N as |k|→∞ for any N > 0 and hence, F(k) is a good function.

THEOREM 2.4.2 If f(x) is a good function with the Fourier transform (2.4.27), then the inverse Fourier transform is given by

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\,F(k)\,dk. \qquad (2.4.28)$$

PROOF For any > 0, we have

$$\mathscr{F}\left\{e^{-\epsilon x^2}F(-x)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx-\epsilon x^2}\left\{\int_{-\infty}^{\infty} e^{ixt}\,f(t)\,dt\right\}dx.$$

Since f is a good function, the order of integration can be interchanged to obtain

$$\mathscr{F}\left\{e^{-\epsilon x^2}F(-x)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt\int_{-\infty}^{\infty} e^{-i(k-t)x-\epsilon x^2}\,dx$$

Which is, by similar calculation used in Example 2.3.1?

$$= \frac{1}{\sqrt{4\pi\epsilon}}\int_{-\infty}^{\infty}\exp\left[-\frac{(k-t)^2}{4\epsilon}\right]f(t)\,dt\,.$$

Using the fact that

$$\frac{1}{\sqrt{4\pi\epsilon}}\int_{-\infty}^{\infty}\exp\left[-\frac{(k-t)^2}{4\epsilon}\right]dt = 1,$$

We can write

$$\mathscr{F}\left\{e^{-\epsilon x^2}F(-x)\right\} - f(k)\,.\,1$$
$$= \frac{1}{\sqrt{4\pi\epsilon}}\int_{-\infty}^{\infty}[f(t)-f(k)]\exp\left[-\frac{(k-t)^2}{4\epsilon}\right]dt. \quad (2.4.29)$$

Since f is a good function, we have

$$\left|\frac{f(t)-f(k)}{t-k}\right| \le \max_{x\in\mathbb{R}}|f'(x)|\,.$$

It follows from (2.4.29) that

$$\left|\mathscr{F}\left\{e^{-\epsilon x^2}F(-x)\right\} - f(k)\right|$$
$$\le \frac{1}{\sqrt{4\pi\epsilon}}\max_{x\in\mathbb{R}}|f'(x)|\int_{-\infty}^{\infty}|t-k|\,\exp\left[-\frac{(t-k)^2}{4\epsilon}\right]dt$$
$$= \frac{1}{\sqrt{4\pi\epsilon}}\max_{x\in\mathbb{R}}|f'(x)|\;4\epsilon\int_{-\infty}^{\infty}|\alpha|\;e^{-\alpha^2}d\alpha \to 0$$

as → 0, where α $\alpha = \frac{t-k}{2\sqrt{\epsilon}}$.

$$f(k) = \mathscr{F}\{F(-x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}F(-x)\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}F(x)\,dx$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}dx\int_{-\infty}^{\infty}e^{-i\xi x}f(\xi)\,d\xi.$$

This simplifies to the Fourier integral formula (2.2.4) when the variable k is replaced with the variable x, and the theorem is proven.

Example 2.4.1 The Fourier transform of a constant function c is

$$\mathscr{F}\{c\} = \sqrt{2\pi}.c.\delta(k). \quad (2.4.30)$$

In the ordinary sense

$$\mathscr{F}\{c\} = \frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}\,dx$$

Is not a well defined (divergent) integral However, treated as a generalized function, c = c I(x) and we consider $\left\{\exp\left(-\frac{x^2}{4n}\right)\right\}$ as an equivalent sequence to the unit function, I(x). Thus,

$$\mathscr{F}\left\{c\exp\left(-\frac{x^2}{4n}\right)\right\} = \frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-ikx - \frac{x^2}{4n}\right)dx$$

Which is, by Example 2.3.1,

$$\begin{aligned} &= c\sqrt{2n}\exp(-nk^2) = \sqrt{2\pi}.c.\sqrt{\frac{n}{\pi}}\exp(-nk^2) \\ &= \sqrt{2\pi}.c.\delta_n(k) = \sqrt{2\pi}.c.\delta(k) \quad \text{as} \quad n \to \infty, \end{aligned}$$

Since $\{\delta_n(k)\} = \left\{\sqrt{\frac{n}{\pi}}\exp\left(-nk^2\right)\right\}$ is a sequence equivalent to the delta function defined by (2.4.10).

Example 2.4.2 Show that

$$\mathscr{F}\{e^{-ax}H(x)\} = \frac{1}{\sqrt{2\pi}(ik+a)}, \quad a > 0. \qquad (2.4.31)$$

We have, by definition,

$$\mathscr{F}\{e^{-ax}H(x)\} = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\exp\{-x(ik+a)\}dx = \frac{1}{\sqrt{2\pi}(ik+a)}.$$

Example 2.4.3 By considering the function (see Figure 2.5)

Figure out what is the Fourier transform of sgn (x). As seen in Figure 2.5, the vertical axis (y-axis) represents the function fa(x), while the horizontal axis represents the x-axis.

We have, by definition,

$$\mathscr{F}\{f_a(x)\} = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\{(a-ik)x\}dx$$
$$+\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \exp\{-(a+ik)x\}dx$$
$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{a+ik} - \frac{1}{a-ik}\right] = \sqrt{\frac{2}{\pi}} \cdot \frac{(-ik)}{a^2+k^2}.$$

In the limit as a → 0, fa(x) → sgn(x) and then

$$\mathscr{F}\{\mathrm{sgn(x)}\} = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\mathrm{ik}}.$$
$$\mathscr{F}\left\{\sqrt{\frac{\pi}{2}} i\,\mathrm{sgn(x)}\right\} = \frac{1}{k}.$$

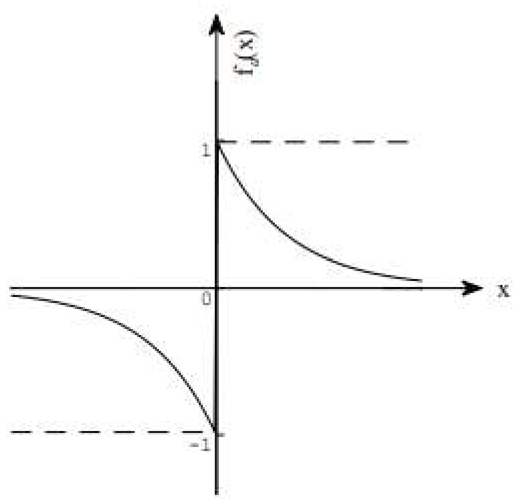


Figure 2.5 Graph of the function fa(x).

Example 2.4.4 (Fourier Integral Theorem)

Using the delta function representation (2.4.12) of a continuous function f(x), we give a short proof of the Fourier integral theorem (2.2.4). We have, by (2.4.12) and (2.4.19),

$$f(x) = \int_{-\infty}^{\infty} f(\xi)\delta(x-\xi)d\xi$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)d\xi \int_{-\infty}^{\infty} e^{ik(x-\xi)}dk$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\left\{\int_{-\infty}^{\infty} e^{-ik\xi}f(\xi)d\xi\right\}dk. \qquad (2.4.33)$$

2.5 BASIC PROPERTIES OF FOURIER TRANSFORMS

THEOREM 2.5.1 If F{f(x)} = F(k), then

(a) (Shifting)	$\mathscr{F}\{f(x-a)\} = e^{-ika}F(k)$,	(2.5.1)
(b) (Scaling)	$\mathscr{F}\{f(ax \pm b)\} = \frac{1}{\lvert a\rvert} e^{\pm\frac{ibk}{a}} F(\frac{k}{a})$, $\quad a \neq 0$	(2.5.2)
(c) (Conjugate)	$\mathscr{F}\{\overline{f(-x)}\} = \overline{\mathscr{F}\{f(x)\}}$,	(2.5.3)
(d) (Translation)	$\mathscr{F}\{e^{iax}f(x)\} = F(k-a)$,	(2.5.4)
(e) (Duality)	$\mathscr{F}\{F(x)\} = f(-k)$,	(2.5.5)
(f) (Composition)	$\int_{-\infty}^{\infty} F(k)g(k)e^{ikx}dk = \int_{-\infty}^{\infty} f(\xi)G(\xi-x)d\xi$, where $G(k) = \mathscr{F}\{g(x)\}$,	(2.5.6)
(g) (Modulation)	$\mathscr{F}\{f(x)\cos ax\} = \frac{1}{2}[F(k-a) + F(k+a)]$ $\mathscr{F}\{f(x)\sin ax\} = \frac{1}{2i}[F(k-a) - F(k+a)]$.	

PROOF (a) We obtain, from the definition

$$\mathscr{F}\{f(x-a)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}f(x-a)dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ik(\xi+a)}f(\xi)d\xi, \qquad (x-a=\xi)$$
$$= e^{-ika}\mathscr{F}\{f(x)\}.$$

The proofs of results (b)–(d) follow easily from the definition of the Fourier transform. We give a proof of the duality (e) and composition (f). We have, by definition,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} F(k) dk = \mathscr{F}^{-1}\{F(k)\}.$$

Interchanging x and k, and then replacing k by −k, we obtain

$$f(-k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} F(x) dx = \mathscr{F}\{F(x)\}.$$

To prove (f), we have

$$\begin{aligned}\int_{-\infty}^{\infty} F(k) g(k)\, e^{ikx} dk &= \int_{-\infty}^{\infty} g(k)\, e^{ikx} dk \;\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik\xi} f(\xi) d\xi \\ &= \int_{-\infty}^{\infty} f(\xi) d\xi \;\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik(\xi - x)} g(k) dk \\ &= \int_{-\infty}^{\infty} f(\xi) G(\xi - x) d\xi.\end{aligned}$$

In particular, when $x = 0$, (2.5.6) reduces to

$$\int_{-\infty}^{\infty} F(k) g(k) dk = \int_{-\infty}^{\infty} f(\xi) G(\xi) d\xi.$$

This is known as the composition rule which can readily be proved.

THEOREM 2.5.2 If f(x) is piecewise continuously differentiable and absolutely integrable, then

(i) F(k) is bounded for $-\infty < k < \infty$,

(ii) F(k) is continuous for $-\infty < k < \infty$.

PROOF It follows from the definition that

$$|F(k)| \le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-ikx}||f(x)|dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|dx = \frac{c}{\sqrt{2\pi}},$$

Where $c = \int_{-\infty}^{\infty} |f(x)|dx =$ constant This proves result (i).

To prove (ii), we have

$$|F(k+h) - F(k)| \le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-ihx} - 1||f(x)|dx$$

$$\le \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} |f(x)|dx.$$

Since lim $|e^{-ihx} - 1|$ 0 for all x ∈ R, we obtain h→0

$$\lim_{h\to 0} |F(k+h) - F(k)| \le \lim_{h\to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-ihx} - 1||f(x)|dx = 0.$$

This shows that F(k) is continuous.

THEOREM 2.5.3 (Riemann-Lebesgue Lemma).

If $F(k) = \mathscr{F}\{f(x)\}$, then

$$\lim_{|k|\to\infty} |F(k)| = 0. \qquad (2.5.7)$$

PROOF Since $e^{-ikx} = -e^{-ikx-i\pi}$, we have

$$F(k) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik(x+\frac{\pi}{k})} f(x)dx$$
$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f\left(x-\frac{\pi}{k}\right) dx.$$

Hence

$$F(k) = \frac{1}{2}\left\{\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{-ikx} f(x)dx - \int_{-\infty}^{\infty} e^{-ikx} f\left(x-\frac{\pi}{k}\right) dx\right]\right\}$$
$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\left[f(x) - f\left(x-\frac{\pi}{k}\right)\right] dx.$$

Therefore

$$|F(k)| \leq \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \left|f(x) - f\left(x-\frac{\pi}{k}\right)\right| dx.$$

Thus, we obtain

$$\lim_{|k|\to\infty} |F(k)| \leq \frac{1}{2\sqrt{2\pi}} \lim_{|k|\to\infty} \int \left|f(x) - f\left(x-\frac{\pi}{k}\right)\right| dx = 0$$

THEOREM 2.5.4 If f(x) is continuously differentiable and f(x) → 0 as x|→∞, then

$$\mathscr{F}\{f'(x)\} = (ik)\mathscr{F}\{f(x)\} = ik\,F(k). \qquad (2.5.8)$$

PROOF We have, by definition,

$$\mathscr{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f'(x)dx$$

Which is, integrating by parts,

$$= \frac{1}{\sqrt{2\pi}}\left[f(x)e^{-ikx}\right]_{-\infty}^{\infty} + \frac{ik}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)dx$$
$$= (ik)F(k).$$

If f(x) is continuously n-times differentiable and f(k) (x) → 0 as |x|→∞ for k = 1, 2,...,(n − 1), then the Fourier transform of the nth derivative is

$$\mathscr{F}\{f^{(n)}(x)\} = (ik)^n \mathscr{F}\{f(x)\} = (ik)^n F(k). \qquad (2.5.9)$$

A repeated application of Theorem 2.5.4 to higher derivatives gives the result

It is possible to get operational findings for partial derivatives of a function of two or more independent variables that are similar to those obtained in (2.58 and 2.59). When u(x,t) is a function of the variables space and time, it is said to be in the domain of u(x,t).

$$\mathscr{F}\left\{\frac{\partial u}{\partial x}\right\} = ik\,U(k,t), \quad \mathscr{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -k^2\,U(k,t),$$
$$\mathscr{F}\left\{\frac{\partial u}{\partial t}\right\} = \frac{dU}{dt}, \quad \mathscr{F}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = \frac{d^2U}{dt^2},$$

Where U (k, t) = F {u(x, t)}.

DEFINITION 2.5.1 The convolution of two integrable functions f(x) and g(x), denoted by (f ∗ g)(x), is defined by

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi, \qquad (2.5.10)$$

Provided the integral in (2.5.10) exists, where the factor $\frac{1}{\sqrt{2\pi}}$ is a question of personal preference This element is often overlooked in the research of convolution since it has no effect on the characteristics of the convolution. convolution. We will include or exclude the factor $\frac{1}{\sqrt{2\pi}}$ freely in this book.

We give some examples of convolution.

Example 2.5.1 Find the convolution of

(a) f(x) = cos x and g(x) = exp(−a|x|), a> 0,

(b) f(x) = χ[a,b](x) and g(x) = x2,

Where χ[a,b](x) is the characteristic function of the interval [a, b] ⊆ R defined by

$$\chi_{[a,b]}(x) = \begin{Bmatrix} 1, & a \le x \le b \\ 0, & \text{otherwise} \end{Bmatrix}.$$

(a) We have, by definition,

$$\begin{aligned}
(f*g)(x) &= \int_{-\infty}^{\infty} f(x-\xi)\ g(\xi)d\xi = \int_{-\infty}^{\infty} \cos(x-\xi)\ e^{-a|\xi|}d\xi \\
&= \int_{-\infty}^{0} \cos(x-\xi)\ e^{a\xi}d\xi + \int_{0}^{\infty} \cos(x-\xi)\ e^{-a\xi}d\xi \\
&= \int_{0}^{\infty} \cos(x+\xi)\ e^{-a\xi}d\xi + \int_{0}^{\infty} \cos(x-\xi)\ e^{-a\xi}d\xi \\
&= 2\cos x \int_{0}^{\infty} \cos\xi\ e^{-a\xi}d\xi \ = \ \frac{2a\cos x}{(1+a^2)}.
\end{aligned}$$

If a = 1, then f ∗ g)(x) = f(x) so that g becomes an identity element of convolution. The question is whether it is true for all g(x).

(b) We have

$$\begin{aligned}
(f*g)(x) &= \int_{-\infty}^{\infty} f(x-\xi)\ g(\xi)d\xi = \int_{-\infty}^{\infty} \chi_{[a,b]}(x-\xi)\ g(\xi)d\xi \\
&= \int_{-\infty}^{\infty} \chi_{[a,b]}(\xi)\ g(x-\xi)d\xi = \int_a^b g(x-\xi)\ d\xi = \int_a^b (x-\xi)^2\ d\xi \\
&= \frac{1}{3}\left\{(x-a)^3 - (x-b)^3\right\}.
\end{aligned}$$

THEOREM 2.5.5 (Convolution Theorem).

If F{f(x)} = F(k) and F{g(x)} = G(k), then

$$\mathscr{F}\{f(x)*g(x)\}=F(k)G(k). \qquad (2.5.11)$$

Or

$$f(x)*g(x)=\mathscr{F}^{-1}\{F(k)G(k)\}, \qquad (2.5.12)$$

or, equivalently,

$$\int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi=\int_{-\infty}^{\infty} e^{ikx}F(k)G(k)dk. \qquad (2.5.13)$$

PROOF We have, by the definition of the Fourier transform,

$$\begin{aligned}\mathscr{F}\{f(x)*g(x)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}dx\int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik\xi}g(\xi)d\xi\int_{-\infty}^{\infty} e^{-ik(x-\xi)}f(x-\xi)dx\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik\xi}g(\xi)d\xi\int_{-\infty}^{\infty} e^{-ik\eta}f(\eta)d\eta=G(k)F(k).\end{aligned}$$

Where, in this proof, the factor $\frac{1}{\sqrt{2\pi}}$ The definition of the convolution includes all essential interchanges of the order of integration, and all necessary interchanges of the order of integration are valid. This brings the proof to a close.

The convolution has the following algebraic properties:

$$\begin{aligned} f*g=g*f \quad &\text{(Commutative)}, & (2.5.14)\\ f*(g*h)=(f*g)*h \quad &\text{(Associative)}, & (2.5.15)\\ (\alpha f+\beta g)*h=\alpha(f*h)+\beta(g*h) \quad &\text{(Distributive)}, & (2.5.16)\\ f*\sqrt{2\pi}\delta=f=\sqrt{2\pi}\delta*f \quad &\text{(Identity)}, & (2.5.17)\end{aligned}$$

Where α and β are constants.

We give proofs of (2.5.15) and (2.5.16). If f ∗ (g ∗ h) exists, then

$$[f*(g*h)](x)=\int_{-\infty}^{\infty} f(x-\xi)(g*h)(\xi)d\xi$$
$$=\int_{-\infty}^{\infty} f(x-\xi)\int_{-\infty}^{\infty} g(\xi-t)h(t)\,dt\;\;d\xi$$

$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x-\xi)\,g(\xi-t)d\xi\right]h(t)dt$$
$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x-t-\eta)\;\;g(\eta)d\eta\right]h(t)dt\quad(\text{put }\xi-t=\eta)$$
$$=\int_{-\infty}^{\infty}[(f*g)(x-t)]\,h(t)dt$$
$$=[(f*g)*h](x).$$

Whereas, in the preceding argument, the exchange of the order of integration may be justified with the assumption of appropriate assumptions To demonstrate (2.5.16), we use the right-hand side of (2.5.16), that is, we prove

$$\alpha(f*h)+\beta(g*h)=\alpha\int_{-\infty}^{\infty} f(x-\xi)h(\xi)d\xi+\beta\int_{-\infty}^{\infty} g(x-\xi)h(\xi)d\xi$$
$$=\int_{-\infty}^{\infty}[\alpha f(x-\xi)+\beta g(x-\xi)]\,h(\xi)d\xi$$
$$=[(\alpha f+\beta g)*h](x).$$

Another demonstration of the convolution's associative property is provided in the next section. On the left-hand side of (2.5.15), we use the Fourier transform to get the result we want, and then we employ the convolution theorem (2.5.5) to get what we want.

$$\mathscr{F}\{f*(g*h)\}=\mathscr{F}\{f\}\,\mathscr{F}\{(g*h)\}$$
$$=F(k)\,[\mathscr{F}\{g\}\,\mathscr{F}\{h\}]$$
$$=F(k)\,[G(k)H(k)]$$
$$=[F(k)G(k)]\,H(k)$$
$$=\mathscr{F}\{(f*g)\}\,\mathscr{F}\{h\}$$
$$=\mathscr{F}\{(f*g)*h\}.$$

Applying the F−1 on both sides, we obtain

$$f * (g * h) = (f * g) * h.$$

Similarly, the convolution theorem 2.5.5 may be used to show all of the characteristics of convolution (2.5.14–2.5.17) with relative ease. As a result of the convolution's commutative feature (2.5.14), the expression (2.5.13) may be expressed as

$$\int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi = \int_{-\infty}^{\infty} e^{ikx}F(k)G(k)dk. \qquad (2.5.18)$$

This is valid for all real x, and hence, putting x = 0 gives

$$\int_{-\infty}^{\infty} f(\xi)g(-\xi)d\xi = \int_{-\infty}^{\infty} f(x)g(-x)dx = \int_{-\infty}^{\infty} F(k)G(k)dk. \qquad (2.5.19)$$

We substitute $g(x) = \overline{f(-x)}$ to obtain

$$G(k) = \mathscr{F}\{g(x)\} = \mathscr{F}\left\{\overline{f(-x)}\right\} = \overline{\mathscr{F}\{f(x)\}} = \overline{F(k)}.$$

Evidently, (2.5.19) becomes

$$\int_{-\infty}^{\infty} f(x)\,\overline{f(x)}dx = \int_{-\infty}^{\infty} F(k)\,\overline{F(k)}dk \qquad (2.5.20)$$

Or

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(k)|^2 dk. \qquad (2.5.21)$$

This is well known as Parseval's relation. For square integrable functions f(x) and g(x), the inner product f, gis defined by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}dx \qquad (2.5.22)$$

So the norm f2 is defined by

$$\|f\|_2^2 = \langle f, f\rangle = \int_{-\infty}^{\infty} f(x)\,\overline{f(x)}dx = \int_{-\infty}^{\infty} |f(x)|^2 dx. \qquad (2.5.23)$$

In the case of all complex-valued Lebesgue square integrable functions with the inner product specified by (2.5.22), the function space L2(R) is a complete normed space with the norm as defined by (2.5.22). (2.5.23). In terms of the norm, the Parseval relation has the following representation:

$$\|f\|_2 = \|F\|_2 = \|\mathscr{F}f\|_2. \qquad (2.5.24)$$

This means that the Fourier transform action is unitary. Physically, the quantity f2 is a measure of energy and F2 represents the power spectrum of f.

THEOREM 2.5.6 (General Parseval’s Relation).

If F{f(x)} = F(k) and F{g(x)} = G(k) then

$$\int_{-\infty}^{\infty} f(x)\,\overline{g(x)}dx = \int_{-\infty}^{\infty} F(k)\,\overline{G(k)}dk. \qquad (2.5.25)$$

PROOF We proceed formally to obtain

$$\begin{aligned}\int_{-\infty}^{\infty} F(k)\,\overline{G(k)}dk &= \int_{-\infty}^{\infty} dk \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iky} f(y)\ dy\,\overline{\int_{-\infty}^{\infty} e^{-ikx} g(x)\ dx}\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\ dy\int_{-\infty}^{\infty}\overline{g(x)}dx\int_{-\infty}^{\infty} e^{ik(x-y)}dk\\ &= \int_{-\infty}^{\infty}\overline{g(x)}\ dx\int_{-\infty}^{\infty}\delta(x-y)f(y)\ dy = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\ dx.\end{aligned}$$

In particular, when g(x) = f(x), the above result agrees with (2.5.20). Second Proof of (2.5.25).

Using the inverse Fourier transform, we have

$$f(x)\overline{g(x)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}F(k)dk\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi x}\,\overline{G(\xi)}\,d\xi$$
$$= \int_{-\infty}^{\infty} F(k)dk\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k-\xi)x}\,\overline{G(\xi)}\,d\xi.$$

Thus

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} F(k)dk\int_{-\infty}^{\infty}\delta(k-\xi)\,\overline{G(\xi)}\,d\xi$$
$$= \int_{-\infty}^{\infty} F(k)\,\overline{G(k)}\,dk.$$

We now use an indirect method to obtain the Fourier transform of sgn(x), that is,

$$\mathscr{F}\{\mathrm{sgn(x)}\} = \sqrt{\frac{2}{\pi}}\,\frac{1}{\mathrm{ik}}. \qquad (2.5.26)$$

From (2.4.26), we find

$$\mathscr{F}\left\{\frac{d}{dx}\mathrm{sgn(x)}\right\} = \mathscr{F}\{2H'(x)\} = 2\mathscr{F}\{\delta(x)\} = \sqrt{\frac{2}{\pi}},$$

Which is, by (2.5.8),

$$ik\,\mathscr{F}\{\mathrm{sgn(x)}\} = \sqrt{\frac{2}{\pi}},$$

Or

$$\mathscr{F}\{\mathrm{sgn(x)}\} = \sqrt{\frac{2}{\pi}}\cdot\frac{1}{\mathrm{ik}}.$$

The Fourier transform of H(x) follows from (2.4.30) and (2.5.26):

$$\mathscr{F}\{H(x)\} = \frac{1}{2}\mathscr{F}\{1+\mathrm{sgn(x)}\} = \frac{1}{2}[\mathscr{F}\{1\} + \mathscr{F}\{\mathrm{sgn(x)}\}]$$
$$= \sqrt{\frac{\pi}{2}}\left[\delta(k) + \frac{1}{i\pi k}\right]. \qquad (2.5.27)$$

2.6 APPLICATIONS OF FOURIER TRANSFORMS TO ORDINARY DIFFERENTIAL EQUATIONS

We consider the nth-order linear ordinary differential equation with constant coefficients

$Ly(x) = f(x),$ ………………..(2.6.1)

Where L is the nth-order differential operator given by

$L \equiv a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0,$ …………………….(2.6.2)

Where an, an−1,...,a1, a0 are constants, D ≡ d dx and f(x) is a given function. Application of the Fourier transform to both sides of (2.6.1) gives

$$[a_n(ik)^n + a_{n-1}(ik)^{n-1} + \cdots + a_1(ik) + a_0]Y(k) = F(k),$$

Where $\mathscr{F}\{y(x)\} = Y(k)$ and $\mathscr{F}\{f(x)\} = F(k).$

Or, equivalently

$$P(ik)Y(k) = F(k),$$

Where

$$P(z) = \sum_{r=0}^{n} a_r z^r.$$

Thus

$$Y(k) = \frac{F(k)}{P(ik)} = F(k)Q(k), \qquad \text{……………………(2.6.3)}$$

Where $Q(k) = \frac{1}{P(ik)}$.

Applying the Convolution Theorem 2.5.5 to (2.6.3) gives the formal solution

$$y(x) = \mathscr{F}^{-1}\{F(k)\,Q(k)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) q(x-\xi) d\xi, \qquad \text{……………………(2.6.4)}$$

Provided q(x) = F−1{Q(k)} is known explicitly.

The differential equation with an abruptly applied impulse function f(x) = x is considered in order to provide a physical explanation of the result (2.6.4).

$$L\{G(x)\} = \delta(x). \qquad \textbf{……………………..(2.6.5)}$$

The solution of this equation can be written from the inversion of (2.6.3) in the form

$$G(x) = \mathscr{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}} Q(k)\right\} = \frac{1}{\sqrt{2\pi}} q(x). \qquad \text{……………………(2.6.6)}$$

Thus, the solution (2.6.4) takes the form

$$y(x) = \int_{-\infty}^{\infty} f(\xi) G(x-\xi) d\xi. \qquad \text{……………………(2.6.7)}$$

Without a doubt, G(x) functions in the same way as a Green's function, that is, it is the reaction to a single unit impulse. In any physical system, f(x) is often referred to as the input function, while y(x) is referred to as the output function derived by the application of the superposition principle. The admittance is the Fourier transform of the function 2G(x) = q(x), where 2G(x) = q(x) is the square root of x. Obtaining the response to a given input function involves determining the Fourier transform of the input function, multiplying the result by the admittance, and then applying the inverse Fourier transform to the product generated from the multiplying and multiplying operations. We will demonstrate these concepts by addressing a simple issue in electrical circuit theory.

Example 2.6.1 (Electric Current in a Simple Circuit).

The current I(t) in a simple circuit containing the resistance R and inductance L satisfies the equation

$$L\frac{dI}{dt} + RI = E(t), \qquad \text{………………….(2.6.8)}$$

Where E(t) is the applied electromagnetic force and R and L are constants. With E(t) = E0 exp(−a|t|), we use the Fourier transform with respect to time t to obtain

$$(ikL + R)\hat{I}(k) = E_0\sqrt{\frac{2}{\pi}}\frac{a}{(a^2 + k^2)}.$$

Or,

$$\hat{I}(k) = \frac{aE_0}{iL}\sqrt{\frac{2}{\pi}}\frac{1}{\left(k - \frac{Ri}{L}\right)(k^2 + a^2)},$$

Where F{I(t)} = ˆI(k). The inverse Fourier transform gives

$$I(t) = \frac{aE_0}{i\pi L}\int_{-\infty}^{\infty}\frac{\exp(ikt)dk}{\left(k - \frac{Ri}{L}\right)(k^2 + a^2)}. \qquad \text{……………….(2.6.9)}$$

This integral can be evaluated by the Cauchy Residue Theorem. For t > 0

$$\begin{aligned} I(t) &= \frac{aE_0}{i\pi L}\cdot 2\pi i\left[\text{Residue at } k = \frac{Ri}{L} + \text{Residue at } k = ia\right] \\ &= \frac{2aE_0}{L}\left[\frac{e^{-\frac{R}{L}t}}{\left(a^2 - \frac{R^2}{L^2}\right)} - \frac{e^{-at}}{2a\left(a - \frac{R}{L}\right)}\right] \\ &= E_0\left[\frac{e^{-at}}{R - aL} - \frac{2aLe^{-\frac{R}{L}t}}{R^2 - a^2L^2}\right]. \end{aligned} \qquad \textbf{………………..(2.6.10)}$$

Similarly, for t < 0, the Residue Theorem gives

$$I(t) = -\frac{aE_0}{i\pi L} \cdot 2\pi i[\text{Residue at } k = -ia]$$
$$= -\frac{2aE_0}{L}\left[\frac{-Le^{at}}{(aL+R)2a}\right] = \frac{E_0 e^{at}}{(aL+R)}.$$ ……………..(2.6 11)

At t = 0, the current is continuous and therefore,

$$I(0) = \lim_{t\to 0} I(t) = \frac{E_0}{R+aL}.$$

If E(t) = δ(t), then Eˆ(k) = $\frac{1}{\sqrt{2\pi}}$ and the solution is obtained by using the inverse Fourier transform

$$I(t) = \frac{1}{2\pi i L}\int_{-\infty}^{\infty} \frac{e^{ikt}}{k - \frac{iR}{L}}\,dk,$$

Which is, by the Theorem of Residues,

$$= \frac{1}{L}[\text{Residue at } k = iR/L]$$
$$= \frac{1}{L}\exp\left(-\frac{Rt}{L}\right).$$ **…………….(2.6.12)**

Thus, the current tends to zero as t → ∞ as expected.

Example 2.6.2 Find the solution of the ordinary differential equation

$$-\frac{d^2u}{dx^2} + a^2u = f(x), \quad -\infty < x < \infty$$ **………….(2.6.13)**

by the Fourier transform method. Application of the Fourier transform to (2.6.13) gives

$$U(k) = \frac{F(k)}{k^2+a^2}.$$

This can readily be inverted by the Convolution Theorem 2.5.5 to obtain

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi,$$

…………..(2.6.14)

Where $g(x) = \mathscr{F}^{-1}\left\{\frac{1}{k^2+a^2}\right\} = \frac{1}{a}\sqrt{\frac{\pi}{2}}$ by Example 2.3.2 Thus, the final solution is

$$u(x) = \frac{1}{2a} \int_{-\infty}^{\infty} f(\xi) e^{-a|x-\xi|} d\xi.$$

………………..(2.6.15)

Example 2.6.3 (The Bernoulli-Euler Beam Equation)

With respect to an infinite beam on an elastic basis, we investigate the vertical deflection u(x) caused by the action of a specified vertical load W. (x). Because of the deflection u(x), the ordinary differential equation is satisfied.

$$EI\frac{d^4u}{dx^4} + \kappa u = W(x), \quad -\infty < x < \infty.$$. ………………………(2.6.16)

where EI is the flexural rigidity and κ is the foundation modulus of the beam. We find the solution assuming that W(x) has a compact support and u, uall tend to zero as |x|→∞.

We first rewrite (2.10.16) as

$$\frac{d^4u}{dx^4} + a^4 u = w(x)$$

……………………..(2.6.17)

Where a4 = κ/EI and w(x) = W(x)/EI. Use of the Fourier transform to (2.6.17) gives

$$U(k) = \frac{W(k)}{k^4 + a^4}.$$

The inverse Fourier transform gives the solution

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{W(k)}{k^4 + a^4} e^{ikx}\, dk$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{k^4 + a^4} dk \int_{-\infty}^{\infty} w(\xi) e^{-ik\xi}\, d\xi$$
$$= \int_{-\infty}^{\infty} w(\xi) G(\xi, x)\, d\xi,$$

……….(2.6.18)

Where

$$G(\xi, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ik(x-\xi)}}{k^4 + a^4} dk = \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos k(x-\xi)\, dk}{k^4 + a^4}.$$

……….(2.10.19)

The integral can be evaluated by the Theorem of Residues or by using the table of Fourier integrals. We simply state the result

$$G(\xi, x) = \frac{1}{2a^3} \exp\left(-\frac{a}{\sqrt{2}}|x-\xi|\right) \sin\left[\frac{a(x-\xi)}{\sqrt{2}} + \frac{\pi}{4}\right]$$

……….(2.6.20)

The explicit answer resulting from a concentrated load of unit strength operating at a position x0 is found by solving for w(x), which is equal to w(0) = (x x0), in particular. As a result, the answer for this situation becomes

$$u(x) = \int_{-\infty}^{\infty} \delta(\xi - x_0) G(x, \xi)\, d\xi = G(x, x_0).$$

…………(2.6.21)

In this way, the kernel G(x,), which is used in the solution (2.10.18), is physically significant in that it represents the deflection as a function of x caused by a unit point load operating at x = 1. As a result, the deflection owing to a point load of strength w(d) at x is equal to w(d) • G(x, x), and hence, (2.10.18) reflects the superposition of all incremental deflections. To provide a more general dynamic problem of an infinite Bernoulli-Euler beam with damping and elastic foundation, the reader is referred to Stadler and Shreeves (1970) and Sheehan and Debnath (1998), who both solved the problem of an infinite Bernoulli-Euler beam with damping and elastic foundation

(1972). For the beam issue, the Fourier-Laplace transform approach was employed to obtain both the steady state and transient solutions, according to these authors.

2.7 SOLUTION OF INTEGRAL EQUATIONS

In order to solve basic integral equations of the convolution type, it is possible to use the Fourier transform approach. Examples are used to demonstrate the process. We begin by solving the Fredholm integral problem using a convolution kernel of the type.

$$\int_{-\infty}^{\infty} f(t)g(x-t)\,dt + \lambda f(x) = u(x),$$ **…………. (2.7.1)**

Where g(x) and u(x) are given functions and λ is a known parameter. Application of the Fourier transform to (2.7.1) gives

$$\sqrt{2\pi}F(k)G(k) + \lambda F(k) = U(k).$$

Or

$$F(k) = \frac{U(k)}{\sqrt{2\pi}G(k)+\lambda}.$$ **. ………..2.7.2**

The inverse Fourier transform leads to a formal solution

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{U(k)e^{ikx}dk}{\sqrt{2\pi}G(k)+\lambda}.$$ **………..2.7.3**

In particular, if g(x) = 1 x so that

$$G(k) = -i\sqrt{\frac{\pi}{2}}\,\mathrm{sgn}\,\mathrm{k},$$

Then the solution becomes

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{U(k)e^{ikx}dk}{\lambda-i\pi\,\mathrm{sgn\,k}}.$$ **……………2.7.4**

If $\lambda=1$ and $g(x)=\frac{1}{2}\left(\frac{x}{|x|}\right)$ so that $G(k)=\frac{1}{\sqrt{2\pi}}\frac{1}{(ik)}$ solution (2.11.3) reduces to the form

$$\begin{aligned}f(x)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(ik)\frac{U(k)e^{ikx}\,dk}{(1+ik)}\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathscr{F}\{u'(x)\}\mathscr{F}\{\sqrt{2\pi}\,e^{-x}\}e^{ikx}\,dk\\&=u'(x)*\sqrt{2\pi}\,e^{-x}=\int_{-\infty}^{\infty}u'(\xi)\exp(\xi-x)\,d\xi.\end{aligned}$$ **………2.7.5**

Example 2.7.1 Find the solution of the integral equation

$$\int_{-\infty}^{\infty}f(x-\xi)f(\xi)\,d\xi=\frac{1}{x^2+a^2}.$$ **…………2.7.6**

Application of the Fourier transform gives

$$\sqrt{2\pi}F(k)\,F(k)=\sqrt{\frac{\pi}{2}}\,\frac{e^{-a|k|}}{a}.$$

Or

$$F(k)=\frac{1}{\sqrt{2a}}\exp\left\{-\frac{1}{2}a|k|\right\}.$$ **…………2.7.7**

The inverse Fourier transform gives the solution

$$\begin{aligned}f(x)&=\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2a}}\int_{-\infty}^{\infty}\exp\left(ikx-\frac{1}{2}a|k|\right)\,dk\\&=\frac{1}{2\sqrt{\pi a}}\left[\int_{0}^{\infty}\exp\left\{-k\left(\frac{a}{2}+ix\right)\right\}\,dk+\int_{0}^{\infty}\exp\left\{-k\left(\frac{a}{2}-ix\right)\right\}\,dk\right]\\&=\frac{1}{2\sqrt{\pi a}}\left[\frac{4a}{(4x^2+a^2)}\right]=\sqrt{\frac{a}{\pi}}\cdot\frac{2}{(4x^2+a^2)}.\end{aligned}$$

Using the table B-1 of Fourier transform (see No. 4), we also get the same result :

$$f(x) = \mathscr{F}^{-1}\{F(k)\} = \sqrt{\frac{a}{x}}\frac{2}{4x^2+a^2}.$$

Example 2.7.2 Solve the integral equation

$$\int_{-\infty}^{\infty} \frac{f(t)\,dt}{(x-t)^2+a^2} = \frac{1}{(x^2+b^2)},\ b>a>0.$$ **…………..2.7.8**

Taking the Fourier transform, we obtain

$$\sqrt{2\pi}\,F(k)\,\mathscr{F}\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}\frac{e^{-b|k|}}{b},$$

Or

$$\sqrt{2\pi}\,F(k)\sqrt{\frac{\pi}{2}}\cdot\frac{e^{-a|k|}}{a} = \sqrt{\frac{\pi}{2}}\frac{e^{-b|k|}}{b}.$$

Thus,

$$F(k) = \frac{1}{\sqrt{2\pi}}\left(\frac{a}{b}\right)\exp\{-|k|(b-a)\}.$$ **……….2.7.9**

The inverse Fourier transform leads to the solution

$$\begin{aligned} f(x) &= \frac{a}{2\pi b}\int_{-\infty}^{\infty}\exp[ikx-|k|(b-a)]dk \\ &= \frac{a}{2\pi b}\left[\int_0^{\infty}\exp[-k\{(b-a)+ix\}]dk + \int_0^{\infty}\exp[-k\{(b-a)-ix\}]\right]dk \\ &= \frac{a}{2\pi b}\left[\frac{1}{(b-a)+ix}+\frac{1}{(b-a)-ix}\right] \\ &= \left(\frac{a}{\pi b}\right)\frac{(b-a)}{(b-a)^2+x^2}. \end{aligned}$$ (2.11.1 **……..(2.7.10)**

Example 2.11.3 Solve the integral equation

$$f(x)+4\int^{\infty} e^{-a|x-t|}f(t)dt=g(x).$$ **……….2.7.11**

Application of the Fourier transform gives

$$F(k)+4\sqrt{2\pi}F(k)\cdot\frac{2a}{\sqrt{2\pi}(a^2+k^2)}=G(k)$$
$$F(k)=\frac{(a^2+k^2)}{a^2+k^2+8a}G(k).$$ **……2.7.12**

The inverse Fourier transform gives

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{(a^2+k^2)G(k)}{a^2+k^2+8a}e^{ikx}dk.$$ **…………2.7.13**

In particular, if a = 1 and g(x) = e−|x| so that G(k) = $=\sqrt{\frac{2}{\pi}}\frac{1}{1+k^2},$ then solution (2.11.13) becomes

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}}{k^2+3^2}dk.$$ **……….2.7.14**

For x > 0, we use a semicircular closed contour in the lower half of the complex plane to evaluate (2.11.14). It turns out that

$$f(x)=\frac{1}{3}e^{-3x}.$$ **…………2.7.15**

Similarly, for x < 0, a semicircular closed contour in the upper half of the complex plane is used to evaluate (2.11.14) so that

$$f(x)=\frac{1}{3}e^{3x},\quad x<0.$$ **…………..2.7.16**

Thus, the final solution is

$$f(x) = \frac{1}{3}\exp(-3|x|).$$ **………..2.7.17**

2.8 SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

In this section, we demonstrate how the Fourier transform method may be utilised to achieve the solution to boundary value and starting value issues for linear partial differential equations of various forms using the Fourier transform technique.

Example 2.12.1 (Dirichlet's Problem in the Half-Plane) We consider the solution of the Laplace equation in the half-plane

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y \geq 0,$$ **…………(2.8.1)**

With the boundary conditions

$$u(x,0) = f(x), \qquad -\infty < x < \infty,$$ **…….2.8.2**

$$u(x,y) \to 0 \text{ as } |x| \to \infty, \quad y \to \infty.$$ **…………(2.8.3)**

We introduce the Fourier transform with respect to x

$$U(k,y) = \frac{1}{\sqrt{2\pi}} \int^{\infty} e^{-ikx} u(x,y)dx$$ **……………(2.8.4)**

So that (2.8.1)–(2.8.3) becomes

$$\frac{d^2U}{dy^2} - k^2U = 0,$$ **…………2.8.5**

$$U(k,0) = F(k), \quad U(k,y) \to 0 \qquad \text{as } y \to \infty.$$ **………………(2.8.6)**

Thus, the solution of this transformed system is

$$U(k,y)=F(k)e^{-|k|y}.$$ **……………(2.8.7)**

Application of the Convolution Theorem 2.5.5 gives the solution

$$u(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi,$$ **…………..(2.8.8)**

Where

$$g(x)=\mathscr{F}^{-1}\{e^{-|k|y}\}=\sqrt{\frac{2}{\pi}}\frac{y}{(x^2+y^2)}.$$ **…………(2.8.9)**

Consequently, the solution (2.8.8) becomes

$$u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)d\xi}{(x-\xi)^2+y^2}, \quad y>0.$$ **……………(2.8.10)**

This is the well-known Poisson integral formula in the half-plane. It is noted that

$$\lim_{y\to 0+} u(x,y)=\int_{-\infty}^{\infty} f(\xi)\left[\lim_{y\to 0+}\frac{y}{\pi}\cdot\frac{1}{(x-\xi)^2+y^2}\right]d\xi=\int_{-\infty}^{\infty} f(\xi)\delta(x-\xi)d\xi,$$ **…………(2.8.11)**

$$\delta(x-\xi)=\lim_{y\to 0+}\frac{y}{\pi}\cdot\frac{1}{(x-\xi)^2+y^2}.$$ **…………….(2.8.12)**

This may be recognized as a solution of the Laplace equation for a dipole source at (x, y)=(ξ, 0).

In particular, when

$$f(x)=T_0H(a-|x|),$$ **……………(2.8.13)**

The solution (2.8.10) reduces to

$$u(x,y) = \frac{yT_0}{\pi}\int_{-a}^{a}\frac{d\xi}{(\xi - x)^2 + y^2}$$
$$= \frac{T_0}{\pi}\left[\tan^{-1}\left(\frac{x+a}{y}\right) - \tan^{-1}\left(\frac{x-a}{y}\right)\right]$$
$$= \frac{T_0}{\pi}\tan^{-1}\left(\frac{2ay}{x^2+y^2-a^2}\right).$$ **…………….(2.8.14)**

The curves in the upper half-plane for which the steady state temperature is constant are known as isothermal curves. In this case, these curves represent a family of circular arcs

$x^2 + y^2 - \alpha y = a^2$ **…………. (2.8.15)**

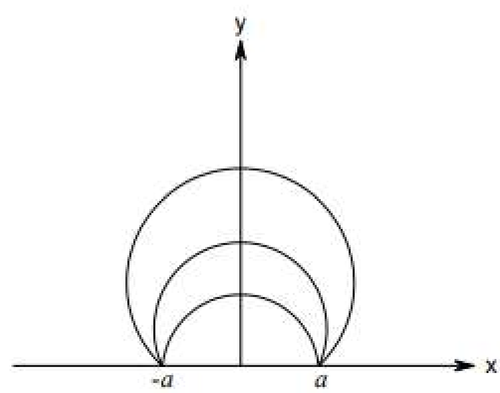


Figure 2.9 A family of circular arcs.

With centers on the y-axis and the fixed end points on the x-axis at x = ±a. The graphs of the arcs are displayed in Figure 2.9. Another special case deals with

$f(x) = \delta(x).$ **………….(2.8.16)**

The solution for this case follows from (2.8.10) and is

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\delta(\xi)d\xi}{(x-\xi)^2+y^2} = \frac{y}{\pi}\frac{1}{(x^2+y^2)}.$$ **…………..(2.8.17)**

Further, we can readily deduce the solution of the Neumann problem in the half-plane from the solution of the Dirichlet problem.

Example 2.8.2 (Neumann's Problem in the Half-Plane) Find a solution of the Laplace equation

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0, \quad \text{…………..} \ \textbf{(2.8.18)}$$

With the boundary condition

$$u_y(x, 0) = f(x), \quad -\infty < x < \infty. \quad \text{………….} \ \textbf{(2.8.19)}$$

This condition indicates the normal derivative on the boundary, and it represents the fluid flow or heat flux at the boundary in terms of physics. It is defined as

We define a new function υ(x, y) = uy(x, y) so that

$$u(x, y) = \int^{y} v(x, \eta)\, d\eta, \quad \text{…………..} \ \textbf{(2.8.20)}$$

Where an arbitrary constant can be added to the right-hand side. Clearly, the function υ satisfies the Laplace equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} = \frac{\partial}{\partial y}(u_{xx} + u_{yy}) = 0,$$

With the boundary condition

$$v(x, 0) = u_y(x, 0) = f(x) \text{ for } -\infty < x < \infty.$$

Thus, υ(x, y) satisfies the Laplace equation with the Dirichlet condition on the boundary. Obviously, the solution is given by (2.12.10); that is,

$$v(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)\, d\xi}{(x - \xi)^2 + y^2}. \quad \text{………..} \ \textbf{(2.8.21)}$$

Then the solution u(x, y) can be obtained from (2.12.20) in the form

$$u(x,y) = \int^{y} v(x,\eta)d\eta = \frac{1}{\pi}\int^{y} \eta\, d\eta \int_{-\infty}^{\infty} \frac{f(\xi)d\xi}{(x-\xi)^2+\eta^2}$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)d\xi \int^{y} \frac{\eta\, d\eta}{(x-\xi)^2+\eta^2}, \quad y>0$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\log[(x-\xi)^2+y^2]d\xi,$$

………. (2.8.22)

In this case, an arbitrary constant may be used to modify the answer. With this in mind, the solution to every Neumann problem is uniquely determined up to an arbitrary constant, as shown in Figure 1.

As an example, see Example 2.8.3. (The Cauchy Problem for the Diffusion Equation). Specifically, we are interested in the initial value issue for a one-dimensional diffusion equation with no sources or sinks.

$$u_t = \kappa u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$ **…………. (2.8.23)**

Where κ is diffusivity constant with the initial condition

$$u(x,0) = f(x), \quad -\infty < x < \infty.$$ **……….(2.8.24)**

We solve this problem using the Fourier transform in the space variable x defined by (2.12.4). Application of this transform to (2.12.23)–(2.12.24) gives

$$U_t = -\kappa k^2 U, \quad t > 0,$$ **………… (2.8.25)**

$$U(k,0) = F(k).$$ **……….. (2.8.26)**

The solution of the transformed system is

$$U(k,t) = F(k)\, e^{-\kappa k^2 t}.$$ **………… (2.8.27)**

The inverse Fourier transform gives the solution

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) \exp[(ikx - \kappa k^2 t)] dk$$

Which is, by the Convolution Theorem 2.5.5?

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi,$$ **………….. (2.8.28)**

Where

$$g(x) = \mathscr{F}^{-1}\{e^{-\kappa k^2 t}\} = \frac{1}{\sqrt{2\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right), \text{ by } (2.3.5).$$

Thus, solution (2.12.28) becomes

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} f(\xi) \exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right] d\xi.$$ **…………… (2.8.29)**

The integrand involved in the solution consists of the initial value f(x) and Green's function (or, elementary solution) G(x − ξ, t) of the diffusion equation for the infinite interval:

$$G(x-\xi,t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right].$$ **………… (2.8.30)**

So, in terms of G(x − ξ, t), solution (2.8.29) can be written as

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi) G(x-\xi,t) d\xi$$ **………….. (2.8.31)**

So that, in the limit as t → 0+, this formally becomes

$$u(x,0) = f(x) = \int_{-\infty}^{\infty} f(\xi) \lim_{t\to 0+} G(x-\xi,t) d\xi.$$

The limit of G(x − ξ, t) represents the Dirac delta function

$$\delta(x-\xi) = \lim_{t \to 0+} \frac{1}{2\sqrt{\pi \kappa t}} \exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right].$$ **…………. (2.8.32)**

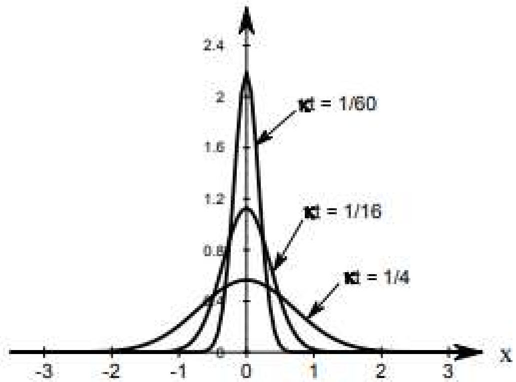


Figure 2.10 Graphs of G(x, t) against x.

Figure 2.10 depicts graphs of G(x, t) for a variety of different values of the time constant. Remember that the integrand in (2.12.31) is composed of the initial temperature distribution f(x) and Green's function G(x t), which represents the temperature response along the rod at time t as a result of an initial unit impulse of heat occurring when the x = t is equal to the initial unit impulse of heat. Physicists interpret the answer (2.12.31) as implying that the original temperature distribution f(x) is divided into a spectrum of impulses of magnitude f() at each point x = in order to generate the final temperature distribution f(x)G(x t). As a consequence, the resultant temperature is integrated to arrive at a solution (2.12.31). The concept of integral superposition is used to describe this. We make the necessary changes to the variable.

2.9 FOURIER COSINE AND SINE TRANSFORMS WITH EXAMPLES

The Fourier cosine integral formula (2.2.8) leads to the Fourier cosine transform and its inverse defined by

$$\mathscr{F}_c\{f(x)\} = F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos kx \, f(x) dx,$$ **………….2.9.1**

$$\mathscr{F}_c^{-1}\{F_c(k)\} = f(x) = \sqrt{\frac{2}{\pi}} \int^\infty \cos kx \, F_c(k) dk,$$ **…………..2.9.2**

Where Fc denotes the Fourier cosine transform operator and F1 c denotes the Fourier cosine transform inverse operator. Furthermore, the Fourier sine integral formula (2.2.9) leads to the Fourier sine transform and its inverse, which are defined as and respectively.

$$\mathscr{F}_s\{f(x)\} = F_s(k) = \sqrt{\frac{2}{\pi}} \int^{\infty} \sin kx\, f(x)dx, \qquad \text{..........2.9.3}$$

$$\mathscr{F}_s^{-1}\{F_s(k)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin kx\ F_s(k)dk, \qquad \text{............2.9.4}$$

Where Fs is the Fourier sine transform operator and F−1 s is its inverse.

Example 2.13.1 Show that

$$\text{(a) } \mathscr{F}_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{a}{(a^2+k^2)}, \quad (a>0). \qquad \text{.........2.9.5}$$

$$\text{(b) } \mathscr{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{k}{(a^2+k^2)}, \quad (a>0). \qquad \text{..........2.9.6}$$

We have

$$\mathscr{F}_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos kx\, dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^{\infty} [e^{-(a-ik)x} + e^{-(a+ik)x}]dx$$

$$\mathscr{F}_c\{e^{-ax}\} = \frac{1}{2}\sqrt{\frac{2}{\pi}} \left[\frac{1}{a-ik} + \frac{1}{a+ik}\right] = \sqrt{\frac{2}{\pi}} \frac{a}{(a^2+k^2)}.$$

The evidence for the second finding is similar and is thus left to the discretion of the reader. Using the preceding findings in conjunction with the Fourier cosine and sine inverse transformations, as well as an interchange of variables, we discover that

$$\mathscr{F}_c\left\{\frac{1}{(x^2+a^2)}\right\} = \sqrt{\frac{\pi}{2}}\,\frac{e^{-ak}}{a},$$
$$\mathscr{F}_s\left\{\frac{x}{(x^2+a^2)}\right\} = \sqrt{\frac{\pi}{2}}\,e^{-ak}.$$

According to the Fourier cosine and sine inverse transformations, we write

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos kx}{k^2+a^2}\,dk = \frac{2}{\pi}\int_0^\infty \frac{k\sin kx}{k^2+a^2}\,dk,\ \ a>0.$$

Interchanging x and k, these results become

$$e^{-ak} = \frac{2a}{\pi}\int_0^\infty \frac{\cos kx}{x^2+a^2}\,dx = \frac{2}{\pi}\int_0^\infty \frac{x\sin kx}{x^2+a^2}\,dx.$$

Thus, it follows that

$$\mathscr{F}_c\left\{\frac{1}{(x^2+a^2)}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos kx}{x^2+a^2}\,dx = \sqrt{\frac{2}{\pi}}\,\frac{\pi}{2a}\,e^{-ak} = \sqrt{\frac{\pi}{2}}\,\frac{e^{-ak}}{a},$$
$$\mathscr{F}_s\left\{\frac{x}{(x^2+a^2)}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x\sin kx}{x^2+a^2}\,dx = \sqrt{\frac{2}{\pi}}\,\frac{\pi}{2}\,e^{-ak} = \sqrt{\frac{\pi}{2}}\,e^{-ak}.$$

Example 2.9.2 Show that'

$$\mathscr{F}_s^{-1}\left\{\frac{1}{k}\exp(-sk)\right\} = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{s}\right). \quad \textbf{..........2.9.7}$$

We have the standard definite integral

$$\sqrt{\frac{\pi}{2}}\mathscr{F}_s^{-1}\{\exp(-sk)\} = \int_0^\infty \exp(-sk)\sin kx\,dk = \frac{x}{s^2+x^2}. \quad \textbf{.........2.9.8}$$

Integrating both sides with respect to s from s to ∞ gives

$$\int_0^\infty \frac{e^{-sk}}{k}\sin kx\,dk = \int_s^\infty \frac{x\,ds}{x^2+s^2} = \left[\tan^{-1}\frac{s}{x}\right]_s^\infty$$
$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{x}\right) = \tan^{-1}\left(\frac{x}{s}\right).$$

……………2.9.9

Thus

$$\mathscr{F}_s^{-1}\left\{\frac{1}{k}\exp(-sk)\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{k}\exp(-sk)\sin xk\,dk$$
$$= \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{s}\right).$$

Example 2.9.3 Show that

$$\mathscr{F}_s\{erfc(ax)\} = \sqrt{\frac{2}{\pi}}\frac{1}{k}\left[1-\exp\left(-\frac{k^2}{4a^2}\right)\right].$$ **……..2.9.10**

We have

$$\mathscr{F}_s\{erfc(ax)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty erfc(ax)\sin kx\,dx$$
$$= \frac{2\sqrt{2}}{\pi}\int_0^\infty \sin kx\,dx\int_{ax}^\infty e^{-t^2}dt.$$

Interchanging the order of integration, we obtain

$$\mathscr{F}_s\{erf(ax)\} = \frac{2\sqrt{2}}{\pi}\int_0^\infty \exp(-t^2)dt\int_0^{t/a}\sin kx\,dx$$
$$= \frac{2\sqrt{2}}{\pi k}\int_0^\infty \exp(-t^2)\left\{1-\cos\left(\frac{kt}{a}\right)\right\}dt$$
$$= \frac{2\sqrt{2}}{\pi k}\left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}\exp\left(-\frac{k^2}{4a^2}\right)\right].$$

Thus

$$\mathscr{F}_s\{erfc(ax)\} = \sqrt{\frac{2}{\pi}}\frac{1}{k}\left[1 - \exp\left(-\frac{k^2}{4a^2}\right)\right].$$

2.10 PROPERTIES OF FOURIER COSINE AND SINE TRANSFORMS

THEOREM 2.10.1 If Fc{f(x)} = Fc(k) and Fs{f(x)} = Fs(k), then

$$\mathscr{F}_c\{f(ax)\} = \frac{1}{a}F_c\left(\frac{k}{a}\right), \quad a > 0.$$ **……………2.10.1**

$$\mathscr{F}_s\{f(ax)\} = \frac{1}{a}F_s\left(\frac{k}{a}\right), \quad a > 0.$$ **…………2.10.2**

Under appropriate conditions, the following properties also hold:

$$\mathscr{F}_c\{f'(x)\} = k\ F_s(k) - \sqrt{\frac{2}{\pi}}\,f(0),$$ **………….2.10.3**

$$\mathscr{F}_c\{f''(x)\} = -k^2\ F_c(k) - \sqrt{\frac{2}{\pi}}\,f'(0),$$ **…………..2.10.4**

$$\mathscr{F}_s\{f'(x)\} = -k\ F_c(k),$$ **………………..2.10.5**

$$\mathscr{F}_s\{f''(x)\} = -k^2\ F_s(k) + \sqrt{\frac{2}{\pi}}\,k\,f(0).$$ **……………2.10.6**

These results can be generalized for the cosine and sine transforms of higherorder derivatives of a function. They are left as exercises.

THEOREM 2.10.2 (Convolution Theorem for the Fourier Cosine Transform)

If $\mathscr{F}_c\{f(x)\} = F_c(k)$ and $\mathscr{F}_c\{g(x)\} = G_c(k)$, then

$$\mathscr{F}_c^{-1}\{F_c(k)G_c(k)\} = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} f(\xi)[g(x+\xi) + g(|x-\xi|)]d\xi.$$ **………………2.10.7**

Or, equivalently,

$$\int_0^\infty F_c(k)G_c(k)\cos kx\,dk = \frac{1}{2}\int_0^\infty f(\xi)[g(x+\xi)+g(|x-\xi|)]d\xi.$$

…………2.10.8

PROOF using the definition of the inverse Fourier cosine transform, we have

$$\begin{aligned}\mathscr{F}_c^{-1}\{F_c(k)G_c(k)\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(k)G_c(k)\cos kx\,dk \\ &= \left(\frac{2}{\pi}\right)\int_0^\infty G_c(k)\cos kx\,dk\int_0^\infty f(\xi)\cos k\xi\,d\xi.\end{aligned}$$

Hence,

$$\begin{aligned}\mathscr{F}_c^{-1}\{F_c(k)G_c(k)\} &= \left(\frac{2}{\pi}\right)\int_0^\infty f(\xi)d\xi\int_0^\infty \cos kx\cos k\xi\, G_c(k)dk \\ &= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(\xi)d\xi\sqrt{\frac{2}{\pi}}\int_0^\infty[\cos k(x+\xi)+\cos k(|x-\xi|)]G_c(k)dk \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty f(\xi)[g(x+\xi)+g(|x-\xi|)]d\xi,\end{aligned}$$

In this case, the definition of the inverse Fourier cosine transform is used. As shown by the fact that (2.10.7). The proof of Theorem 2.14.2 also has the additional consequence of showing

$$\int_0^\infty F_c(k)G_c(k)\cos kx\,dk = \frac{1}{2}\int_0^\infty f(\xi)[g(x+\xi)+g(|x-\xi|)]d\xi.$$

This proves result (2.10.8). Putting $x = 0$ in (2.10.8), we obtain

$$\int_0^\infty F_c(k)G_c(k)dk = \int_0^\infty f(\xi)g(\xi)d\xi = \int_0^\infty f(x)g(x)dx.$$

Substituting $g(x) = f(x)$ gives, since $Gc(k) = Fc(k)$,

$$\int_0^\infty |F_c(k)|^2 dk = \int_0^\infty |f(x)|^2 dx. \qquad \text{...............(2.10.9)}$$

This is the Parseval relation for the Fourier cosine transform.

Similarly, we obtain

$$\int_0^\infty F_s(k)G_s(k)\cos kx\, dk$$
$$= \sqrt{\frac{2}{\pi}}\int_0^\infty G_s(k)\cos kx\, dk \int_0^\infty f(\xi)\sin k\xi\, d\xi$$

Which is, by interchanging the order of integration?

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(\xi)d\xi \int_0^\infty G_s(k)\sin k\xi \cos kx\, dk$$
$$= \frac{1}{2}\int_0^\infty f(\xi)d\xi \sqrt{\frac{2}{\pi}}\int_0^\infty G_s(k)[\sin k(\xi+x)+\sin k(\xi-x)]dk$$
$$= \frac{1}{2}\int_0^\infty f(\xi)[g(\xi+x)+g(\xi-x)]d\xi,$$

In which the inverse Fourier sine transform is used. Thus, we find

$$\int_0^\infty F_s(k)G_s(k)\cos kx\, dk = \frac{1}{2}\int_0^\infty f(\xi)[g(\xi+x)+g(\xi-x)]d\xi. \qquad \text{...............(2.10.10)}$$

Or, equivalently,

$$\mathscr{F}_c^{-1}\{F_s(k)G_s(k)\} = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(\xi)[g(\xi+x)+g(\xi-x)]d\xi. \qquad \text{.................(2.10.11)}$$

Result (2.10.10) or (2.10.11) is also called the Convolution Theorem of the Fourier cosine transform.

Putting x = 0 in (2.10.10) gives

$$\int_0^\infty F_s(k)G_s(k)dk = \int_0^\infty f(\xi)g(\xi)d\xi = \int_0^\infty f(x)g(x)dx.$$

Replacing g(x) by f(x) gives the Parseval relation for the Fourier sine transform

$$\int_0^\infty |F_s(k)|^2 dk = \int_0^\infty |f(x)|^2 dx.$$

……………….(2.10.12)

2.11 APPLICATIONS OF FOURIER TRANSFORMS IN MATHEMATICAL STATISTICS

Fourier transforms or FourierStieltjes transform of the distribution function of a random variable are both used to determine the characteristic function of a random variable in probability theory and mathematical statistics. Using the methodology of characteristic functions, many significant conclusions in probability theory and mathematical statistics may be achieved, and their proofs can be made more rigorous while maintaining their precision. Consequently, Fourier transformations are significant in probability theory and mathematical statistics, among other areas of study.

DEFINITION 2.11.1 (Distribution Function) The distribution function F(x) of a random variable X is defined as the probability, that is, F(x) = P(X

It is immediately evident from this definition that the distribution function satisfies the following properties:

(i) F(x) is a non-decreasing function, that is, $F(x1) \leq F(x2)$ if $x1 < x2$.

(ii) F(x) is continuous only from the left at a point x, that is, $F(x - 0) = F(x)$, but $F(x + 0) = F(x)$.

(iii) $F(-\infty) = 0$ and $F(+\infty)=1$.

Assuming X is a continuous variable, and assuming there exists a non-negative function f(x) such that the following relationship holds for any real x, then

$$F(x) = \int_{-\infty}^{x} f(x)dx, \qquad \text{...............}\mathbf{(2.11.1)}$$

Where F(x) is the distribution function of the random variable X, then the function f(x) is called the probability density or simply the density function of the random variable X.

It is immediately obvious that every density function f(x) satisfies the following properties:

(i) $$F(+\infty) = \int_{-\infty}^{\infty} f(x)dx = 1. \qquad \text{...............(2.11.2)}$$

(ii) For every real a and b where a<b,

$$P(a \leq X \leq b) = F(b) - F(a) = \int_{a}^{b} f(x)dx. \qquad \text{...........(2.11.3)}$$

(iii) If f(x) is continuous at some point x, then F (x) = f(x).

It is noted that every real function f(x) which is non-negative, and integrable over the whole real line and satisfies (2.17.2ab), is the probability density function of a continuous random variable X. On the other hand, the function F(x) defined by (2.17.1) satisfies all properties of a distribution function.

DEFINITION 2.17.2 (Characteristic Function) If X is a continuous random variable with the density function f(x), then the characteristic function, φ(t) of the random variable X or of the distribution function F(x) is defined by the formula

$$\phi(t) = E(\exp(itX)) = \int_{-\infty}^{\infty} f(x)\exp(itx)dx, \qquad \text{...........}\mathbf{(2.11.4)}$$

where E[g(X)] is called the expected value of the random variable g(X). In problems of mathematical statistics, it is convenient to define the Fourier transform of f(x) and its inverse in a slightly different way by

$$\mathscr{F}\{f(x)\} = \phi(t) = \int_{-\infty}^{\infty} \exp(itx) f(x) dx,$$ **………………..(2.11.5)**

$$\mathscr{F}^{-1}\{\phi(t)\} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-itx) \phi(t) dt.$$ **…………………….(2.11.6)**

Evidently, the characteristic function of F(x) is the Fourier transform of the density function f(x). The Fourier transform of the distribution function follows from the fact that

$$\mathscr{F}\{F'(x)\} = \mathscr{F}\{f(x)\} = \phi(t),$$

Or

$$\mathscr{F}\{F(x)\} = it^{-1}\phi(t).$$ **…………….(2.11.7)**

The composition of two distribution functions F1(x) and F2(x) is defined by

$$F(x) = F_1(x) * F_2(x) = \int_{-\infty}^{\infty} F_1(x-y) F_2'(y) dy.$$ **…………(2.11.8)**

Thus, the Fourier transform of (2.17.7) gives

$$\begin{aligned} i\,t^{-1}\phi(t) &= \mathscr{F}\left\{ \int_{-\infty}^{\infty} F_1(x-y) F_2'(y) dy \right\} \\ &= \mathscr{F}\{F_1(x)\} \mathscr{F}\{f_2(x)\} = it^{-1}\phi_1(t)\phi_2(t), \end{aligned}$$

Whence an important result follows:

$$\phi(t) = \phi_1(t)\phi_2(t),$$ **………..(2.11.9)**

Where φ1(t) and φ2(t) are the characteristic functions of the distribution functions F1(x) and F2(x), respectively.

The nth moment of a random variable X is defined by

$$m_n = E[X^n] = \int^{\infty} x^n f(x)dx, \quad n = 1, 2, 3, \ldots. \quad (2.11.10)$$

Provided this integral exists. The first moment m1 (or simply m) is called the expectation of X and has the form

$$m = E(X) = \int_{-\infty}^{\infty} x\, f(x)dx. \quad (2.11.11)$$

The moment of any order n may be found by calculating the integral of that order (2.17.9). The examination of the integral, on the other hand, is a challenging process in general. It is possible to tackle this challenge with the assistance of the characteristic function defined by (2.17.4). The result of differentiating (2.17.4) n times and setting t = 0 is a reasonably straightforward formula.

$$m_n = \int_{-\infty}^{\infty} x^n f(x)dx = (-i)^n \phi^{(n)}(0), \quad (2.11.12)$$

Where n = 1, 2, 3,....

When n = 1, the expectation of a random variable X becomes

$$m_1 = E(X) = \int_{-\infty}^{\infty} x f(x)dx = (-i)\phi'(0). \quad (2.11.13)$$

As a result, the simple formula (2.17.11) utilising the derivatives of the characteristic function allows for the presence of any arbitrary order as well as the calculation of the moment of that order. For similar reasons, we may express the variance 2 of a random variable as follows in terms of the characteristic function:

$$\sigma^2 = \int_{-\infty}^{\infty} (x-m)^2 f(x)dx = m_2 - m_1^2$$
$$= \{\phi'(0)\}^2 - \phi''(0). \qquad (2.11.14)$$

Example 2.10.1 Find the moments of the normal distribution defined by the density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\}. \qquad (2.11.15)$$

The characteristic function of the normal distribution is the Fourier transform of f(x), which is

$$\phi(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right] dx.$$

We substitute x − m = y and use Example 2.3.1 to obtain

$$\phi(t) = \frac{\exp(itm)}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ity} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy = \exp\left(itm - \frac{1}{2}t^2\sigma^2\right). \qquad (2.11.16)$$

Thus

$$m_1 = (-i)\phi'(0) = m,$$
$$m_2 = -\phi''(0) = (m^2 + \sigma^2),$$
$$m_3 = m(m^2 + 3\sigma^2).$$

Finally, the variance of the normal distribution is

$$m_2 - m_1^2 = \sigma^2. \qquad (2.11.17)$$

It is clear from the above explanation that characteristic functions are very valuable for the exploration of specific issues in mathematical statistics. We will conclude this section by addressing some more characteristics of characteristic functions.

THEOREM 2.11.1 (Addition Theorem).

The characteristic function of the sum of a finite number of independent random variables is equal to the product of their characteristic functions.

PROOF Suppose X1, X2,...,Xn are n independent random variables and Z = X1 + X2 + ⋯ + Xn. Further, suppose φ1(t), φ2(t),...,φn(t), and φ(t) are the characteristic functions of X1, X2,...,Xn and Z, respectively.

$$\phi(t) = E[\exp(itZ)] = E\left[\exp\left\{it\left(X_1 + X_2 + \cdots + X_n\right)\right\}\right],$$

Which is, by the independence of the random variables?

$$= E(e^{itX_1})E(e^{itX_2})\cdots E(e^{itX_n})$$
$$= \phi_1(t)\,\phi_2(t)\cdots\phi_n(t). \qquad (2.11.18)$$

This proves the Addition Theorem.

Example 2.11.2 Find the expected value and the standard deviation of the sum of n independent normal random variables.

Suppose X1, X2,...,Xn are n independent random variables with the normal distributions N(mr, σr), where r = 1, 2,...,n. The respective characteristic functions of these distributions are

$$\phi_r(t) = \exp\left[itm_r - \frac{1}{2}t^2\sigma_r^2\right], \quad r = 1, 2, 3, \ldots, n. \qquad (2.11.19)$$

Because of the independence of X1, X2,...,Xn, the random variable Z = X1 + X2 + ⋯ + Xn has the characteristic function

$$\phi(t) = \phi_1(t)\phi_2(t)\cdots\phi_n(t)$$

$$= \exp\left[it(m_1 + m_2 + \cdots + m_n) - \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2)t^2\right]. \qquad (2.11.20)$$

This represents the characteristic function of the normal distribution N(m1 + $\cdots + m_n, \sqrt{\sigma_1^2 + \cdots + \sigma_n^2})$. Thus, the expected value of Z is (m1 + m2 + ⋯ + mn) and its standard deviation is $(\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2)^{\frac{1}{2}}$.

Finally, we state the fundamental Central Limit Theorems without proof.

THEOREM 2.11.2 (The Le´vy-Crame´r Theorem)

For example, consider the sequence of random variables Xn, where Fn(x) and n(t) represent the distribution and characteristic functions of Xn, respectively. So, for any point t on the real line, the series n(t) is convergent to the distribution function F(x) only if and only if the sequence t is also convergent to a function (t) continuous in some neighbourhood of the origin, in which case the sequence n(t) is nonconverging. Because of this, the limit function n(t) n(t) is equal to the characteristic function of the limit distribution function F(x), and the convergence n(t) n(t) is uniform in every finite interval down the t-axis.

THEOREM 2.11.3 (The Central Limit Theorem in Probability).

Supose f(x) is a nonnegative absolutely integrable function in R and has the following properties:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1, \quad \int_{-\infty}^{\infty} x\,f(x)\,dx = 1, \quad \int_{-\infty}^{\infty} x^2\,f(x)\,dx = 1.$$

If f n = f ∗ f ∗ ... ∗ f is the convolution product of f with itself n times, then

$$\lim_{n\to\infty} \int_{a\sqrt{n}}^{b\sqrt{n}} f^n(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\,dx \quad -\infty < a < b < \infty. \tag{2.11.21}$$

We direct the reader to the next section for a demonstration of the theorem. All of the concepts introduced in this section may be extended to multidimensional distribution functions by the use of multiple Fourier transforms. We recommend Lukacs to any readers who are interested (1960).

CHAPTER 3

LAPLACE TRANSFORMS AND THEIR PROPERTIES

"What we know is not much. What we do not know is immense." Pierre-Simon Laplace

"By concentrating our attention on abstract combinations, the algebraic analysis quickly causes us to lose sight of the fundamental purpose [of our study], and it is only at the conclusion that we are able to return to the original objective." When one abandons oneself to the processes of analysis, however, one is led to the universality of the approach and the inestimable benefit of changing the reasoning by mechanical methods to findings that are often unattainable by geometry... The beauty that results from a lengthy succession of phrases that are related one to the other and all come from a single core notion can be found in no other language.

Pierre-Simon Laplace

"... For Laplace, on the contrary, mathematical analysis was an instrument that he bent to his purposes for the most varied applications, but always subordinating the method itself to the content of each question. Perhaps posterity will ..."

Simeon-Denis Poisson

3.1 INTRODUCTION

Throughout this chapter, we will provide the formal definition of the Laplace transform and demonstrate how to directly compute the Laplace transforms of various simple functions from the formulation. Section 3.3 contains a description of the Laplace transform's existence and nonexistence requirements. We go through the fundamental operational aspects of the Laplace transforms, such as convolution and its properties, as well as the differentiation and integration of Laplace transforms, in considerable depth. Introduction to the inverse Laplace transform is provided in Section 3.7, followed by the development of four techniques of evaluating the inverse transform with examples. The Heaviside Expansion Theorem, as well as the Tauberian theorems for the Laplace transform, are also addressed in detail.

3.2 DEFINITION OF THE LAPLACE INTEGRAL AND EXAMPLES

As a starting point, we will look at the Fourier Integral Formula (2.2.4), which describes the representation of a function defined on the variable x as follows:

$$f_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk \int_{-\infty}^{\infty} e^{-ikt} f_1(t) dt. \qquad (3.2.1)$$

We next set f1(x) ≡ 0 in −∞ <x< 0 and write

$$f_1(x) = e^{-cx} f(x) H(x) = e^{-cx} f(x), \qquad x > 0, \qquad (3.2.2)$$

Where c is a positive fixed number, so that (3.2.1) becomes

$$f(x) = \frac{e^{cx}}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk \int_{0}^{\infty} \exp\{-t(c+ik)\} f(t) dt. \qquad (3.2.3)$$

With a change of variable, c + ik = s, i dk = ds we rewrite (3.2.3) as

$$f(x) = \frac{e^{cx}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp\{(s-c)x\} ds \int_{0}^{\infty} e^{-st} f(t) dt. \qquad (3.2.4)$$

Thus, the Laplace transform of f(t) is formally defined by

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = \int_{0}^{\infty} e^{-st} f(t) dt, \qquad \operatorname{Re} s > 0, \qquad (3.2.5)$$

Where est is the transform kernel and s is the transform variable, which is a complex integer, the transform is defined as Under general circumstances on f(t), its transform f(s) is analytic in s on the half-plane, where Re s>a. This is known as the analytic transformation. The formal definition of the inverse Laplace transform is provided by the following result (3.2.4).

$$\mathscr{L}^{-1}\{\bar{f}(s)\} = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \bar{f}(s)\, ds, \qquad c > 0. \qquad (3.2.6)$$

Obviously, L and L −1 are linear integral operators.

Using the definition (3.2.5), we can calculate the Laplace transforms of some simple and elementary functions.

Example 3.2.1 If f(t) = 1 for t > 0, then

$$\bar{f}(s) = \mathscr{L}\{1\} = \int_0^\infty e^{-st} dt = \frac{1}{s}. \qquad (3.2.7)$$

Example 3.2.2 If f(t) = eat, where a is a constant, then

$$\mathscr{L}\{e^{at}\} = \bar{f}(s) = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}, \qquad s > a. \qquad (3.2.8)$$

Example 3.2.3 If f(t) = sin at, where a is a real constant, then

$$\mathscr{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt = \frac{1}{2i} \int_0^\infty [e^{-t(s-ia)} - e^{-t(s+ia)}] \, dt \qquad (3.2.9)$$
$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right] = \frac{a}{s^2+a^2}.$$

Similarly,

$$\mathscr{L}\{\cos at\} = \frac{s}{s^2+a^2}. \qquad (3.2.10)$$

Example 3.2.4 If f(t) = sinh at or cosh at, where a is a real constant, then

$$\mathscr{L}\{\sinh at\} = \int_0^\infty e^{-st} \sinh at \, dt = \frac{a}{s^2-a^2}, \qquad (3.2.11)$$

$$\mathscr{L}\{\cosh at\} = \int_0^\infty e^{-st} \cosh at \, dt = \frac{s}{s^2-a^2}. \qquad (3.2.12)$$

Example 3.2.5 If f(t) = tn, where n is a positive integer, then

$$\bar{f}(s) = \mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}. \qquad (3.2.13)$$

We recall (3.2.7) and formally differentiate it with respect to s. This gives

$$\int_0^\infty t\, e^{-st}\, dt = \frac{1}{s^2}, \qquad (3.2.14)$$

Which means that

$$\mathscr{L}\{t\} = \frac{1}{s^2}. \qquad (3.2.15)$$

Differentiating (3.2.14) with respect to s gives

$$\mathscr{L}\{t^2\} = \int_0^\infty t^2 e^{-st}\, dt = \frac{2}{s^3}. \qquad (3.2.16)$$

Similarly, differentiation of (3.2.7) n times yields

$$\mathscr{L}\{t^n\} = \int_0^\infty t^n e^{-st}\, dt = \frac{n!}{s^{n+1}}. \qquad (3.2.17)$$

Example 3.2.6

(a) [If a(> −1) is a real number, then]

$$\mathscr{L}\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}}, \qquad (s>0). \qquad (3.2.18)$$

(b) Show that

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{t^{x-1}}{e^t - 1}\, dt$$

(a) We have

$$\mathscr{L}\{t^a\} = \int_0^\infty t^a e^{-st}\, dt,$$

Which is, by putting st = x,

$$= \frac{1}{s^{a+1}} \int_0^\infty x^a e^{-x} dx = \frac{\Gamma(a+1)}{s^{a+1}},$$

Where Γ (a) represents the gamma function defined by the integral

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx, \qquad a > 0. \qquad (3.2.19)$$

It can be shown that the gamma function satisfies the relation

$$\Gamma(a+1) = a\Gamma(a). \qquad (3.2.20)$$

Obviously, the result (3.2.18) is an extension of the previous result (3.2.17). Whenever an is a positive integer, the second conditional is a special case of the first conditional. It is possible to obtain the well-known integral representation of the Riemann Zeta function x(x) from this example, which is denoted by the symbol

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{t^{x-1}}{e^t - 1}\, dt.$$

It follows from (3.2.18) that

$$\frac{\Gamma(x)}{s^x} = \mathscr{L}\{t^{x-1}\} = \int_0^\infty e^{-st}\, t^{x-1}\, dt, \; s > 0.$$

Summing this result over s, we obtain

$$\Gamma(x)\sum_{s=1}^{\infty}\frac{1}{s^x} = \int_0^{\infty} t^{x-1}\sum_{s=1}^{\infty} e^{-st}\,dt$$
$$= \int_0^{\infty} t^{x-1}\frac{e^{-t}}{1-e^{-t}}\,dt = \int_0^{\infty} t^{x-1}\frac{1}{e^t-1}\,dt$$
$$= \int_0^{\infty}\frac{t^{x-1}}{e^t-1}\,dt.$$

If x = 2 $\int_0^{\infty}\frac{t}{e^t-1}\,dt = \Gamma(2)\zeta(2) = \frac{\pi^2}{6}.$

In particular, when a = −1 2 , result (3.2.18) gives

$$\mathscr{L}\left\{\frac{1}{\sqrt{t}}\right\} = \frac{\Gamma(\frac{1}{2})}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}, \quad \text{where} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \qquad (3.2.21)$$

Similarly,

$$\mathscr{L}\left\{\sqrt{t}\right\} = \frac{\Gamma(\frac{3}{2})}{s^{3/2}} = \frac{\sqrt{\pi}}{2}\frac{1}{s^{3/2}}, \qquad (3.2.22)$$

Where

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Example 3.2.7 If $f(t) = erf\left(\frac{a}{2\sqrt{t}}\right)$, then

$$\mathscr{L}\left\{erf\left(\frac{a}{2\sqrt{t}}\right)\right\} = \frac{1}{s}(1-e^{-a\sqrt{s}}), \qquad (3.2.23)$$

where erf(t) is the error function defined by (2.15.13).

To prove (3.2.23), we begin with the definition (3.2.5) so that

$$\mathscr{L}\left\{erf\left(\frac{a}{2\sqrt{t}}\right)\right\}=\int_0^\infty e^{-st}\left[\frac{2}{\sqrt{\pi}}\int_0^{a/2\sqrt{t}} e^{-x^2}dx\right]dt.$$

Which is, by putting $x=\frac{a}{2\sqrt{t}}$ or $t=\frac{a^2}{4x^2}$ and interchanging the order of integration,

$$=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}dx\int_0^{a^2/4x^2} e^{-st}dt$$

$$=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\frac{1}{s}\left\{1-\exp\left(-\frac{a^2s}{4x^2}\right)\right\}dx$$

$$=\frac{1}{s}\cdot\frac{2}{\sqrt{\pi}}\left[\int_0^\infty e^{-x^2}dx-\int_0^\infty \exp\left\{-\left(x^2+\frac{sa^2}{4x^2}\right)\right\}dx\right],$$

Where the integral

$$\int_0^\infty \exp\left\{-\left(x^2+\frac{\alpha^2}{x^2}\right)\right\}dx=\frac{1}{2}\left[\int_0^\infty\left(1-\frac{\alpha}{x^2}\right)\exp\left[-\left(x+\frac{\alpha}{x}\right)^2+2\alpha\right]\right.$$
$$\left.+\int_0^\infty\left(1+\frac{\alpha}{x^2}\right)\exp\left[-\left(x-\frac{\alpha}{x}\right)^2-2\alpha\right]\right]dx,$$

Which is, by putting y = $\left(x\pm\frac{\alpha}{x}\right)$, $dy=\left(1\mp\frac{\alpha}{x^2}\right)$ dx, and observing that the first integral vanishes,

$$=\frac{1}{2}e^{-2\alpha}\int_{-\infty}^\infty e^{-y^2}dy=\frac{\sqrt{\pi}}{2}e^{-2\alpha},\qquad \alpha=\frac{a\sqrt{s}}{2}.$$

Consequently,

$$\mathscr{L}\left\{erf\left(\frac{a}{2\sqrt{t}}\right)\right\}=\frac{1}{s}\frac{2}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2}-\frac{\sqrt{\pi}}{2}e^{-a\sqrt{s}}\right]=\frac{1}{s}[1-e^{-a\sqrt{s}}].$$

We use (3.2.23) to find the Laplace transform of the complementary error function defined by (2.15.14) and obtain

$$\mathscr{L}\left\{erfc\left(\frac{a}{2\sqrt{t}}\right)\right\}=\frac{1}{s}\,e^{-a\sqrt{s}}. \qquad (3.2.24)$$

The proof of this result follows from erfc(x)=1 − erf (x) and L {1} = 1 s .

Example 3.2.8 If f(t) = J0(at) is a Bessel function of order zero, then

$$\mathscr{L}\left\{J_0(at)\right\}=\frac{1}{\sqrt{s^2+a^2}}. \qquad (3.2.25)$$

Using the series representation of J0(at), we obtain

$$\mathscr{L}\left\{J_0(at)\right\}=\mathscr{L}\left[1-\frac{a^2t^2}{2^2}+\frac{a^4t^4}{2^2\cdot4^2}-\frac{a^6t^6}{2^2\cdot4^2\cdot\cdot6^2}+\cdots\right]$$

$$=\frac{1}{s}-\frac{a^2}{2^2}\frac{2!}{s^3}+\frac{a^4}{2^2\cdot4^2}\cdot\frac{4!}{s^5}-\frac{a^6}{2^2\cdot4^2\cdot6^2}\cdot\frac{6!}{s^7}+\cdots$$
$$=\frac{1}{s}\left[1-\frac{1}{2}\left(\frac{a^2}{s^2}\right)+\frac{1\cdot3}{2\cdot4}\left(\frac{a^4}{s^4}\right)-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{a^6}{s^6}\right)+\cdots\right]$$
$$=\frac{1}{s}\left[\left(1+\frac{a^2}{s^2}\right)^{-\frac{1}{2}}\right]=\frac{1}{\sqrt{a^2+s^2}}.$$

Similarly, using J0(t) = −J1(t) gives

$$\mathscr{L}\left\{J_1(at)\right\}=\frac{1}{a}\left[\frac{s}{\sqrt{s^2+a^2}}-1\right]. \qquad (3.2.26)$$

Example 3.2.9 The Laplace transform of the Gaussian function. Show that

$$\mathscr{L}\left\{e^{-a^2t^2}\right\}=\frac{\sqrt{\pi}}{2a}\,e^{\frac{s^2}{4a^2}}\operatorname{erfc}\left(\frac{s}{2a}\right),\quad a>0, \qquad (3.2.27)$$

We have, by definition,

$$\mathscr{L}\left\{e^{-a^2t^2}\right\} = \int_0^\infty e^{-a^2t^2-st}\,dt = e^{\frac{s^2}{4a^2}}\int_0^\infty e^{-a^2\left(1+\frac{s}{2a^2}\right)^2}\,dt$$
$$= e^{\frac{s^2}{4a^2}}\int_{\frac{s^2}{2a^2}}^\infty e^{-a^2u^2}\,du = \frac{1}{a}\,e^{\frac{s^2}{4a^2}}\int_{\frac{s}{2a}}^\infty e^{-a^2u^2}\,du$$
$$= \frac{\sqrt{\pi}}{2a}\,e^{\frac{s^2}{4a^2}}\,\mathrm{erfc}\left(\frac{s}{2a}\right).$$

When $a = \frac{1}{\sqrt{2}}$, $\mathscr{L}\left\{e^{-\frac{t^2}{2}}\right\} = \sqrt{2\pi}\,e^{\frac{s^2}{2}}\,\mathrm{erfc}\left(\frac{s}{\sqrt{2}}\right)$.

3.3 EXISTENCE CONDITION FOR THE LAPLACE TRANSFORM

An exponentially increasing function f(t) on the interval 0 t is said to be of order a(>0) on the interval when there exists a positive constant K such that for any t>T

$$|f(t)| \le Ke^{at}, \qquad (3.3.1)$$

And we write this symbolically as

$$f(t) = O(e^{at}) \quad \text{as} \quad t \to \infty. \qquad (3.3.2)$$

Or, equivalently,

$$\lim_{t\to\infty} e^{-bt}|f(t)| \le K \lim_{t\to\infty} e^{-(b-a)t} = 0, \quad b > a. \qquad (3.3.3)$$

Such a function f(t) is simply called an exponential order as t → ∞, and clearly, it does not grow faster than Keat as t → ∞.

THEOREM 3.3.1

If a function f(t) is continuous or piecewise continuous in every finite interval (0, T), and of exponential order eat, then the Laplace transform of f(t) exists for all s provided Re s>a.

PROOF

We have

$$|\bar{f}(s)| = \left|\int_0^\infty e^{-st} f(t)dt\right| \leq \int_0^\infty e^{-st}|f(t)|dt \qquad (3.3.4)$$

$$\leq K \int_0^\infty e^{-t(s-a)}dt = \frac{K}{s-a}, \text{ for Re } s > a.$$

As a result, the evidence is conclusive. Notably, the criteria specified in Theorem 3.3.1 are adequate rather than required conditions, as is seen in Figure 3. lims | f(s)| = 0, which is equivalent to lims f(s) = 0. It also follows from (3.3.4) that lims f(s) = 0. The Laplace transform's limiting property might be thought of as the effect of this. In contrast, the Laplace transform of any continuous (or piecewise continuous) function is not f(s) = s or s2. This is due to the fact that f(s) does not go to zero as the number of terms in the function increases. Furthermore, even if a function f(t) = exp(at2), a > 0 is continuous, it cannot be transformed using the Laplace transform since it is not of the exponential order.

$$\lim_{t\to\infty} \exp(at^2 - st) = \infty.$$

3.4 SOME BASIC PROPERTIES OF LAPLACE TRANSFORM

THEOREM 3.4.1 (Heaviside's First Shifting Theorem).

If L {f(t)} = ¯f(s), then

$$\mathscr{L}\{e^{-at} f(t)\} = \bar{f}(s+a), \qquad (3.4.1)$$

Where a is a real constant.

PROOF We have, by definition,

$$\mathscr{L}\{e^{-at} f(t)\} = \int_0^\infty e^{-(s+a)t} f(t)dt = \bar{f}(s+a).$$

Example 3.4.1 The following results readily follow from (3.4.1)

$$\mathscr{L}\{t^n e^{-at}\} = \frac{n!}{(s+a)^{n+1}}, \qquad (3.4.2)$$

$$\mathscr{L}\{e^{-at}\sin bt\} = \frac{b}{(s+a)^2+b^2}, \qquad (3.4.3)$$

$$\mathscr{L}\{e^{-at}\cos bt\} = \frac{s+a}{(s+a)^2+b^2}. \qquad (3.4.4)$$

THEOREM 3.4.2

If L {f(t)} = $\bar{f}(s)$, then the Second Shifting property holds:

$$\mathscr{L}\{f(t-a)H(t-a)\} = e^{-as}\bar{f}(s) = e^{-as}\mathscr{L}\{f(t)\}, \qquad a>0. \qquad (3.4.5)$$

Or, equivalently,

$$\mathscr{L}\{f(t)H(t-a)\} = e^{-as}\mathscr{L}\{f(t+a)\}. \qquad (3.4.6)$$

Where $H(t-a)$ is the Heaviside unit step function defined by (2.3.9).

It follows from the definition that

$$\mathscr{L}\{f(t-a)H(t-a)\} = \int_0^\infty e^{-st} f(t-a)H(t-a)dt$$
$$= \int_a^\infty e^{-st} f(t-a)dt,$$

Which is, by putting $t - a = \tau$,

$$= e^{-sa}\int_0^\infty e^{-s\tau} f(\tau)d\tau = e^{-sa}\bar{f}(s).$$

We leave it to the reader to prove (3.4.6). In particular, if $f(t) = 1$, then

$$\mathscr{L}\{H(t-a)\} = \frac{1}{s}\exp(-sa). \qquad (3.4.7)$$

Example 3.4.2 Use the shifting property (3.4.5) or (3.4.6) to find the Laplace transform of

(a) $f(t)=\begin{Bmatrix} 1, & 0<t<1 \\ -1, & 1<t<2 \\ 0, & t>2 \end{Bmatrix}$, (b) $g(t)=\sin t\, H(t-\pi)$.

To find L {f(t)}, we write f(t) as

$$f(t)=1-2\,H(t-1)+H(t-2).$$

Hence

$$\begin{aligned}\bar{f}(s)=\mathscr{L}\{f(t)\} &= \mathscr{L}\{1\}-2\,\mathscr{L}\{H(t-1)\}+\mathscr{L}\{H(t-2)\} \\ &= \frac{1}{s}-\frac{2\,e^{-s}}{s}+\frac{e^{-2s}}{s}.\end{aligned}$$

To obtain L {g(t)}, we use (3.4.6) so that

$$\bar{g}(s)=\mathscr{L}\{\sin t\, H(t-\pi)\} = -e^{-\pi s}\,\mathscr{L}\{\cos t\} = -\frac{s\,e^{-\pi s}}{s^2+1}.$$

Scaling Property:

$$\mathscr{L}\{f(at+b)\}=\frac{1}{a}\,e^{\frac{sb}{a}}\,\bar{f}\left(\frac{s}{a}\right), \quad a>0. \qquad (3.4.8)$$

Example 3.4.3 Show that the Laplace transform of the square wave function f(t) defined by

$$f(t)=H(t)-2H(t-a)+2H(t-2a)-2H(t-3a)+\cdots \qquad (3.4.9)$$

Is

$$\bar{f}(s)=\frac{1}{s}\tanh\left(\frac{as}{2}\right). \qquad (3.4.10)$$

The graph of f(t) is shown in Figure 3.1.

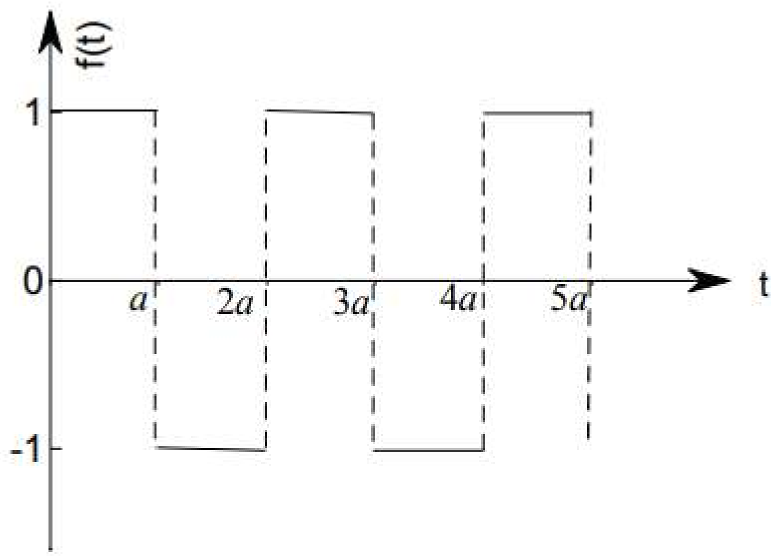


Figure 3.1 Square wave function

$$f(t) = H(t) - 2H(t-a) = 1 - 2\cdot 0 = 1, \qquad 0 < t < a$$
$$f(t) = H(t) - 2H(t-a) + 2H(t-2a)$$
$$= 1 - 2\cdot 1 + 2\cdot 0 = -1, \qquad 0 < a < t < 2a.$$

Thus,

$$\bar{f}(s) = \frac{1}{s} - 2\cdot\frac{e^{-as}}{s} + 2\cdot\frac{e^{-2as}}{s} - 2\cdot\frac{e^{-3as}}{s} + \cdots$$
$$= \frac{1}{s}[1 - 2r(1 - r + r^2 - \cdots)], \qquad \text{where} \quad r = e^{-as}$$
$$= \frac{1}{s}\left[1 - \frac{2r}{1+r}\right] = \frac{1}{s}\left[1 - \frac{2e^{-as}}{1+e^{-as}}\right]$$
$$= \frac{1}{s}\left(\frac{1-e^{-as}}{1+e^{-as}}\right) = \frac{1}{s}\left(\frac{e^{\frac{as}{2}} - e^{-\frac{as}{2}}}{e^{\frac{as}{2}} + e^{-\frac{as}{2}}}\right) = \frac{1}{s}\tanh\left(\frac{as}{2}\right).$$

Example 3.4.4 (The Laplace Transform of a Periodic Function If f(t) is a periodic function of period a, and if L {f(t)} exists, show that

$$\mathscr{L}\{f(t)\} = [1 - \exp(-as)]^{-1} \int_0^a e^{-st} f(t)\,dt. \qquad (3.4.11)$$

We have, by definition,

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^a e^{-st} f(t)dt + \int_a^\infty e^{-st} f(t)dt.$$

Letting t = τ + a in the second integral gives

$$\bar{f}(s) = \int_0^a e^{-st} f(t)dt + \exp(-sa) \int_0^\infty e^{-s\tau} f(\tau + a)d\tau,$$

Which is, due to f(τ + a) = f(τ) and replacing the dummy variable τ by t in the second integral,

$$= \int_0^a e^{-st} f(t)dt + \exp(-sa) \int_0^\infty e^{-st} f(t)dt.$$

Finally, combining the second term with the left-hand side, we obtain (3.4.11). In particular, we calculate the Laplace transform of a rectified sine wave, that is, f(t) = |sin at|. This is a periodic function with period $\frac{\pi}{a}$. We have

$$\int_0^{\frac{\pi}{a}} e^{-st} \sin at\, dt = \left[\frac{e^{-st}(-a\cos at - s\sin at)}{(s^2 + a^2)}\right]_0^{\frac{\pi}{a}} = \frac{a\left\{1 + \exp\left(-\frac{s\pi}{a}\right)\right\}}{(s^2 + a^2)}.$$

Clearly, the property (3.4.11) gives

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \frac{a}{(s^2 + a^2)} \cdot \frac{1 + \exp\left(-\frac{s\pi}{a}\right)}{1 - \exp\left(-\frac{s\pi}{a}\right)} \\
&= \frac{a}{(s^2 + a^2)} \left[\frac{\exp\left(\frac{s\pi}{2a}\right) + \exp\left(-\frac{s\pi}{2a}\right)}{\exp\left(\frac{s\pi}{2a}\right) - \exp\left(-\frac{s\pi}{2a}\right)}\right] \\
&= \frac{a}{s^2 + a^2} \coth\left(\frac{\pi s}{2a}\right).
\end{aligned}$$

Using Example 3.4.4, we can solve Example 3.4.3. Clearly, f(t) in (3.4.9) is a periodic function of period 2a so that

$$\begin{aligned}\bar{f}(s) &= \frac{1}{1-e^{-2as}}\int_0^{2a} f(t)\,e^{-st}\,dt \\ &= \frac{1}{1-e^{-2as}}\left[\int_0^{a} e^{-st}\,dt - \int_a^{2a} e^{-st}\,dt\right]\,dt \\ &= \frac{1}{1-e^{-2as}}\left[\frac{1}{s}\left(1-e^{-sa}+e^{-2as}-e^{-sa}\right)\right]\,dt \\ &= \frac{1}{s\left(1-e^{-2as}\right)}\left(1-e^{-sa}\right)^2 = \frac{1}{s}\left(\frac{1-e^{-sa}}{1+e^{-sa}}\right) \\ &= \frac{1}{s}\left(\frac{e^{\frac{sa}{2}}-e^{-\frac{sa}{2}}}{e^{\frac{sa}{2}}+e^{-\frac{sa}{2}}}\right) = \frac{1}{s}\tanh\left(\frac{as}{2}\right).\end{aligned}$$

THEOREM 3.4.3 (Laplace Transforms of Derivatives)

If $\mathscr{L}\{f(t)\} = \bar{f}(s)$, then

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0) = s\bar{f}(s) - f(0), \qquad (3.4.12)$$

$$\mathscr{L}\{f''(t)\} = s^2\mathscr{L}\{f(t)\} - sf(0) - f'(0) = s^2\bar{f}(s) - s\,f(0) - f'(0). \quad (3.4.13)$$

More generally,

$$\mathscr{L}\{f^{(n)}(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0),$$

where $f^{(r)}(0)$ is the value of $f^{(r)}(t)$ at $t=0$, $r=0,1,\ldots,(n-1)$.

PROOF we have, by definition,

$$\mathscr{L}\{f'(t)\} = \int_0^{\infty} e^{-st} f'(t)dt,$$

Which is, integrating by parts

$$= \left[e^{-st} f(t)\right]_0^\infty + s \int_0^\infty e^{-st} f(t) dt$$
$$= s\bar{f}(s) - f(0),$$

In which we assumed f(t) e−st → 0 as t → ∞. Similarly,

$$\begin{aligned} \mathscr{L}\{f''(t)\} &= s\mathscr{L}\{f'(t)\} - f'(0), \qquad \text{by (3.4.12)} \\ &= s[s\ \bar{f}(s) - f(0)] - f'(0) \\ &= s^2 \bar{f}(s) - s f(0) - f'(0), \end{aligned}$$

Where we have assumed e−stf (t) → 0 as t → ∞

A approach similar to this may be used to demonstrate the overall conclusion (3.4.14). Notably, when the Laplace transform is used to the partial derivatives of a function of two or more independent variables, the results are quite similar to those obtained above. u(x,t) is a function of the two variables x and t, for example, and it is defined as

$$\mathscr{L}\left\{\frac{\partial u}{\partial t}\right\} = s\bar{u}(x,s) - u(x,0), \qquad (3.4.15)$$

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2 \bar{u}(x,s) - s\ u(x,0) - \left[\frac{\partial u}{\partial t}\right]_{t=0}, \qquad (3.4.16)$$

$$\mathscr{L}\left\{\frac{\partial u}{\partial x}\right\} = \frac{d\bar{u}}{dx}, \quad \mathscr{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2 \bar{u}}{dx^2}. \qquad (3.4.17)$$

As a result of the results (3.4.12) to (3.4.14), it can be concluded that the Laplace transform converts the differentiation process into an algebraic operation. As a result, the Laplace transform may be utilised to solve ordinary or partial differential equations with high efficacy in this context.

Example 3.4.5 Use (3.4.14) to find $\mathscr{L}\{t^n\}$.

Here $f(t) = t^n$, $f'(t) = nt^{n-1}, \cdots, f^{(n)}(t) = n!$ and $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. Thus,

$$\mathscr{L}\{n!\} = \mathscr{L}\left\{f^{(n)}(t)\right\} = s^n \mathscr{L}\{f(t)\} = s^n \mathscr{L}\{t^n\}.$$

Or,

$$\mathscr{L}\{t^n\} = \frac{n!}{s^n}\mathscr{L}\{1\} = \frac{n!}{s^{n+1}}.$$

3.5 THE CONVOLUTION THEOREM AND PROPERTIES OF CONVOLUTION

THEOREM 3.5.1 (Convolution Theorem).

If $\mathscr{L}\{f(t)\} = \bar{f}(s)$ and $\mathscr{L}\{g(t)\} = \bar{g}(s)$, then

$$\mathscr{L}\{f(t)*g(t)\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = \bar{f}(s)\bar{g}(s). \qquad (3.5.1)$$

Or, equivalently,

$$\mathscr{L}^{-1}\{\bar{f}(s)\bar{g}(s)\} = f(t)*g(t), \qquad (3.5.2)$$

Where f(t)∗g(t) is called the convolution of f(t) and g(t) and is defined by the integral

$$f(t)*g(t) = \int_0^t f(t-\tau)g(\tau)d\tau. \qquad (3.5.3)$$

The integral in (3.5.3) is often referred to as the convolution integral (or Faltung) and is denoted simply by (f ∗ g)(t).

PROOF We have, by definition,

$$\mathscr{L}\{f(t)*g(t)\} = \int_0^\infty e^{-st}dt\int_0^t f(t-\tau)g(\tau)d\tau, \qquad (3.5.4)$$

Figure 3.2 depicts the zone of integration in the t plane, where the region of integration is as depicted. The integration in (3.5.4) is done first with respect to t from t = 0 to t = t of the vertical strip and then with respect to t from t = 0 to t of the vertical strip by moving the vertical strip from t = 0 outwards to cover the whole area beneath the line t. After that, we reverse the order of integration such that we integrate first along the horizontal strip from t = 0 to t = 1, and then along the horizontal strip from t = 0 to t = 0 by shifting the horizontal strip vertically from t = 0 to t = 1. As a result, (3.5.4) is transformed into

$$\mathscr{L}\{f(t)*g(t)\} = \int_0^\infty g(\tau)d\tau \int_{t=\tau}^\infty e^{-st} f(t-\tau)dt,$$

Which is, by the change of variable $t - \tau = x$,

$$\begin{aligned}\mathscr{L}\{f(t)*g(t)\} &= \int_0^\infty g(\tau)d\tau \int_0^\infty e^{-s(x+\tau)} f(x)dx \\ &= \int_0^\infty e^{-s\tau} g(\tau)d\tau \int_0^\infty e^{-sx} f(x)dx = \bar{g}(s)\,\bar{f}(s).\end{aligned}$$

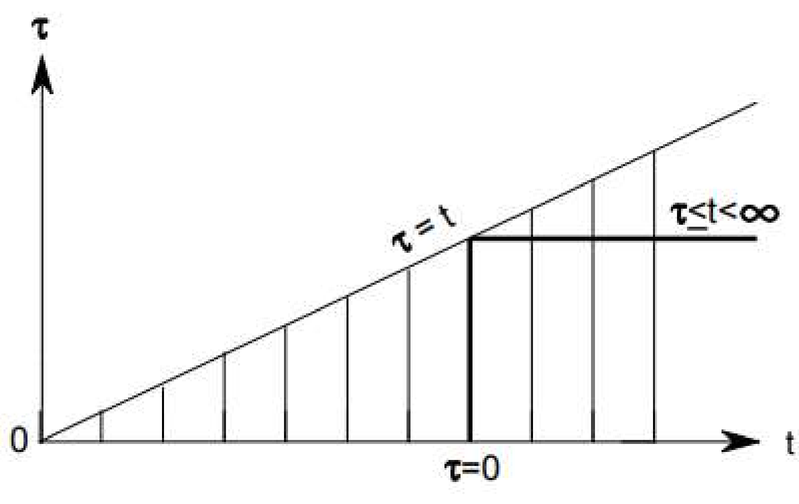


Figure 3.2 Region of integration.

This completes the proof.

PROOF (Second Proof.) We have, by definition,

$$\begin{aligned}\bar{f}(s)\bar{g}(s) &= \int_0^\infty e^{-s\sigma} f(\sigma)d\sigma \int_0^\infty e^{-s\mu} g(\mu)d\mu \\ &= \int_0^\infty \int_0^\infty e^{-s(\sigma+\mu)} f(\sigma)g(\mu)d\sigma\, d\mu, \qquad (3.5.5)\end{aligned}$$

This is shown in Figure 3.3, where the double integral is taken across the whole first quarter R of the plane R, which is bordered by the values of 0 and 0, as seen in Figure 3.3. (a). In the t-plane, we alter the variables = t, and then we change the values of the variables t, and then we change the values of the variables t, and then we change the values of the variables t, and then we change the values of all the variables. As a result, (3.5.5) is transformed into

$$\begin{aligned}\bar{f}(s)\bar{g}(s) &= \int_0^\infty e^{-st}dt \int_{\tau=0}^{\tau=t} f(t-\tau)g(\tau)d\tau \\ &= \mathscr{L}\left\{\int_0^t f(t-\tau)g(\tau)d\tau\right\} \\ &= \mathscr{L}\left\{f(t)*g(t)\right\}.\end{aligned}$$

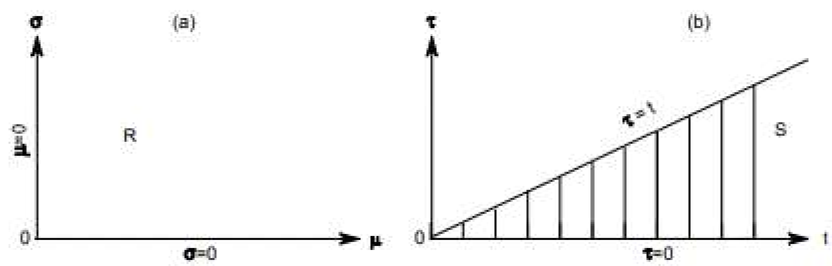


Figure 3.3 Regions of integration.

This completes the second proof.

PROOF (Third Proof.) By definition,

$$\mathscr{L}\left\{f(t)*g(t)\right\} = \int_0^\infty e^{-st}\left[\int_0^t f(t-\tau)g(\tau)d\tau\right]dt.$$

Now using the Heaviside unit step function, we can write

$$\begin{aligned}\mathscr{L}\{f(t)*g(t)\} &= \int_0^\infty e^{-st}dt\int_0^\infty f(t-\tau)H(t-\tau)g(\tau)d\tau\\ &= \int_0^\infty g(\tau)d\tau\int_0^\infty e^{-st}f(t-\tau)H(t-\tau)dt\\ &= \int_0^\infty e^{-s\tau}\bar{f}(s)g(\tau)d\tau \qquad \text{using Theorem 3.4.2}\\ &= \bar{f}(s)\int_0^\infty e^{-s\tau}g(\tau)d\tau = \bar{f}(s)\bar{g}(s).\end{aligned}$$

This proves the theorem.

Note: A more rigorous proof of the convolution theorem can be found in any standard treatise (see Doetsch, 1950) on Laplace transforms. The convolution operation has the following properties:

$$f(t)*\{g(t)*h(t)\} = \{f(t)*g(t)\}*h(t), \qquad \text{(Associative)}, \quad (3.5.6)$$

$$f(t)*g(t) = g(t)*f(t), \qquad \text{(Commutative)}, \quad (3.5.7)$$

$$f(t)*\{ag(t)+bh(t)\} = af(t)*g(t)+bf(t)*h(t), \qquad \text{(Distributive)}, \quad (3.5.8)$$

$$f(t)*\{ag(t)\} = \{af(t)\}*g(t) = a\{f(t)*g(t)\}, \qquad (3.5.9)$$

$$\mathscr{L}\{f_1*f_2*f_3*\cdots*f_n\} = \bar{f}_1(s)\bar{f}_2(s)\cdots\bar{f}_n(s), \qquad (3.5.10)$$

$$\mathscr{L}\{f^{*n}\} = \{\bar{f}(s)\}^n, \qquad (3.5.11)$$

Where a and b are constants. f ∗n = f∗f∗···∗f is sometimes called the nth convolution. Remark: It is obvious from (3.5.6) and (3.5.7) that the set of all Laplace transformable functions is a commutative semigroup with respect to the operation. Because fg1 does not, in general, have a Laplace transform, the collection of all Laplace transformable functions does not constitute a group.

We now prove the associative property. We have

$$f(t)*\{g(t)*h(t)\} = \int_0^t f(\tau)\int_0^{t-\tau} g(t-\sigma-\tau)h(\sigma)d\sigma\; d\tau \qquad (3.5.12)$$

$$= \int_0^t h(\sigma) \int_0^{t-\sigma} g(t-\tau-\sigma) f(\tau) d\tau \, d\sigma$$
$$= h(t) * \{f(t) * g(t)\} = \{f(t) * g(t)\} * h(t). \quad (3.5.13)$$

where (3.5.13) is obtained from (3.5.12) by interchanging the order of integration combined with the fact that $0 \le \sigma \le t - \tau$ and $0 \le \tau \le t$ imply $0 \le \tau \le t - \sigma$ and $0 \le \sigma \le t$. Properties (3.5.10) and (3.5.11) follow immediately from the associative law of the convolution.

To prove (3.5.7), we recall the definition of the convolution and make a change of variable $t - \tau = t$. This gives

$$f(t) * g(t) = \int_0^t f(t-\tau) g(\tau) d\tau = \int_0^t g(t-t') f(t') dt' = g(t) * f(t).$$

The proofs of (3.5.8)–(3.5.9) are very simple and hence, may be omitted.

Example 3.5.1 Obtain the convolutions

(a) $t * e^{at}$, (b) $(\sin at * \sin at)$, (c) $\frac{1}{\sqrt{\pi t}} * e^{at}$,

(d) $1 * \frac{a}{2} \frac{e^{-a^2/4t}}{\sqrt{\pi t^3}}$, (e) $\cos t * e^{2t}$, (f) $t * t * t$.

We have

(a) $t * e^{at} = \int_0^t \tau e^{a(t-\tau)} d\tau = e^{at} \int_0^t \tau e^{-a\tau} d\tau = \frac{1}{a^2}(e^{at} - at - 1)$.

(b) $\sin at * \sin at = \int_0^t \sin a\tau \, \sin a(t-\tau) d\tau = \frac{1}{2a}(\sin at - at \cos at)$.

(c) $\frac{1}{\sqrt{\pi t}} * e^{at} = \frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{\tau}} e^{a(t-\tau)} d\tau$, Which is, by putting $\sqrt{a\tau} = x,$

$$\frac{1}{\sqrt{\pi t}} * e^{at} = \frac{2e^{at}}{\sqrt{\pi a}} \int_0^{\sqrt{at}} e^{-x^2} dx = \frac{e^{at}}{\sqrt{a}} \, erf\left(\sqrt{at}\right).$$

(d) We have

$$1 * \frac{a}{2} \frac{e^{-a^2/4t}}{\sqrt{\pi t^3}} = \frac{a}{2\sqrt{\pi}} \int_0^t \frac{e^{-a^2/4\tau}}{\tau^{3/2}} d\tau,$$

Which is, by letting $\frac{a}{2\sqrt{\tau}} = x,$

$$= \frac{2}{\sqrt{\pi}} \int_{\frac{a}{2\sqrt{t}}}^{\infty} e^{-x^2} dx = erfc\left(\frac{a}{2\sqrt{t}}\right).$$

$$\text{(e)} \ \cos t * e^{2t} = \int_0^t \cos(t-\tau)\, e^{2\tau}\, d\tau = \frac{1}{2}\int_0^t e^{2\tau}\left\{e^{i(t-\tau)} + e^{-i(t-\tau)}\right\} d\tau$$

$$= \left[\frac{e^{i(t-\tau)+2\tau}}{2(2-i)} + \frac{e^{-i(t-\tau)+2\tau}}{2(2+i)}\right] = \frac{2}{5}e^{2t} + \frac{1}{5}(\sin t - 2\cos t)$$

$$\text{(f)} \ (t*t)*t = \left[\int_0^t (t-\tau)\,\tau\, d\tau\right] * t = \frac{1}{6}t^3 * t = \frac{1}{6}\int_0^t (t-\tau)\,\tau^3\, d\tau = \frac{t^5}{5!}.$$

Example 3.5.2 (a) Using the Convolution Theorem 3.5.1, prove that

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \qquad (3.5.14)$$

Where Γ(m) is the gamma function, and B(m, n) is the beta function defined by

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \quad (m>0, n>0). \qquad (3.5.15)$$

(b) Show that

$$t^m * t^n = t^{m+n+1} B(m+1, n+1). \qquad (3.5.16)$$

(a) To prove (3.5.14), we consider

$$f(t) = t^{m-1}\ (m > 0) \quad \text{and} \quad g(t) = t^{n-1},\ (n > 0).$$

Evidently $\bar{f}(s) = \dfrac{\Gamma(m)}{s^m}$ and $\bar{g}(s) = \dfrac{\Gamma(n)}{s^n}$.

We have

$$\begin{aligned} f * g &= \int_0^t \tau^{m-1}(t-\tau)^{n-1} d\tau = \mathscr{L}^{-1}\{\bar{f}(s)\bar{g}(s)\} \\ &= \Gamma(m)\Gamma(n)\mathscr{L}^{-1}\{s^{-(m+n)}\} \\ &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} t^{m+n-1}. \end{aligned}$$

Letting t = 1, we derive the result

$$\int_0^1 \tau^{m-1}(1-\tau)^{n-1} d\tau = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \frac{(m-1)!(n-1)!}{(m+n-1)!},$$

Which proves the result (3.5.14). (b) We have, by Convolution Theorem 3.5.1,

$$\mathscr{L}\{t^m * t^n\} = \mathscr{L}\{t^m\}\mathscr{L}\{t^n\} = \frac{\Gamma(m+1)}{s^{m+1}} \frac{\Gamma(n+1)}{s^{n+1}} = \frac{\Gamma(m+1)\Gamma(n+1)}{s^{m+n+2}}.$$

Using the inverse Laplace transform, we obtain

$$\begin{aligned} t^m * t^n &= \Gamma(m+1)\Gamma(n+1)\mathscr{L}^{-1}\left\{\frac{1}{s^{m+n+2}}\right\} \\ &= \Gamma(m+1)\Gamma(n+1)\frac{t^{m+n+1}}{\Gamma(m+n+2)} \\ &= t^{m+n+1} B(m+1, n+1) \qquad \text{using (3.4.14)} \\ &= t^{m+n+1} \int_0^1 x^m (1-x)^n dx. \end{aligned}$$

3.6 DIFFERENTIATION AND INTEGRATION OF LAPLACE TRANSFORMS

THEOREM 3.6.1

If f(t) = O(eat) as t → ∞, then the Laplace integral

$$\int_0^\infty e^{-st} f(t)dt, \qquad (3.6.1)$$

Is uniformly convergent with respect to t provided s ≥ a1 where a1 > a.

PROOF Since

$$|e^{-st} f(t)| \leq Ke^{-t(s-a)} \leq Ke^{-t(a_1-a)} \quad \text{for all } s \geq a_1$$

And $\int_0^\infty e^{-t(a_1-a)}$ dt exists for a1 > a, by the Weierstrass test, the Laplace integral is uniformly convergent for all s>a1 where a1 > a. This completes the proof.

In view of the uniform convergence of (3.6.1), differentiation of (3.2.5) with respect to s within the integral sign is permissible. Hence,

$$\begin{aligned} \frac{d}{ds}\bar{f}(s) &= \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)dt \\ &= -\int_0^\infty t f(t) e^{-st} dt = -\mathscr{L}\{tf(t)\}. \qquad (3.6.2) \end{aligned}$$

Similarly, we obtain

$$\frac{d^2}{ds^2}\bar{f}(s) = (-1)^2 \mathscr{L}\{t^2 f(t)\}, \qquad (3.6.3)$$

$$\frac{d^3}{ds^3}\bar{f}(s) = (-1)^3 \mathscr{L}\{t^3 f(t)\}. \qquad (3.6.4)$$

More generally,

$$\frac{d^n}{ds^n}\bar{f}(s) = (-1)^n \mathscr{L}\{t^n f(t)\}. \tag{3.6.5}$$

Results (3.6.5) can be stated in the following theorem:

THEOREM 3.6.2 (Derivatives of the Laplace Transform).

If $\mathscr{L}\{f(t)\} = \bar{f}(s)$, then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s), \tag{3.6.6}$$

$$\mathscr{L}\{t^n (f*g)(t)\} = (-1)^n \frac{d^n}{ds^n}\left[\bar{f}(s)\bar{g}(s)\right], \tag{3.6.7}$$

Where n = 0, 1, 2, 3, ...

Example 3.6.1 Show that

(a) $\mathscr{L}\{t^n e^{-at}\} = \dfrac{n!}{(s+a)^{n+1}}$, (b) $\mathscr{L}\{t\cos at\} = \dfrac{s^2-a^2}{(s^2+a^2)^2}$,

(c) $\mathscr{L}\{t\sin at\} = \dfrac{2as}{(s^2+a^2)^2}$, (d) $\mathscr{L}\{t\, f'(t)\} = -\left\{s\dfrac{d}{ds}\bar{f}(s) + \bar{f}(s)\right\}$,

(e) $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$.

(a) Application of Theorem 3.6.2 gives

$$\mathscr{L}\{t^n e^{-at}\} = (-1)^n \frac{d^n}{ds^n}\cdot\frac{1}{(s+a)} = (-1)^{2n}\frac{n!}{(s+a)^{n+1}}.$$

(b)

$$\mathscr{L}\{t\cos at\} = (-1)\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = \frac{s^2-a^2}{(s^2+a^2)^2}.$$

Results (c) and (d) can be proved similarly. (e) Here we have

$$\mathscr{L}\{t^n.1\} = (-1)^n \frac{d^n}{ds^n}.\frac{1}{s} = (-1)^{2n}\frac{n!}{s^{n+1}} = \frac{n!}{s^{n+1}}.$$

THEOREM 3.6.3 (Integral of the Laplace Transform).

If L {f(t)} = ¯f(s), then

$$(a) \qquad \mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\, ds, \qquad (3.6.8)$$

$$(b) \qquad \int_0^\infty \bar{f}(s)\, ds = \int_0^\infty \frac{f(t)}{t}\, dt. \qquad (3.6.9)$$

PROOF (a) In view of the uniform convergence of (3.6.1), ¯f(s) can be integrated with respect to s in (s, ∞) so that

$$\int_s^\infty \bar{f}(s) ds = \int_s^\infty ds \int_0^\infty e^{-st} f(t) dt = \int_0^\infty f(t) dt \int_s^\infty e^{-st} ds$$
$$= \int_0^\infty \frac{f(t)}{t} e^{-st} dt = \mathscr{L}\left\{\frac{f(t)}{t}\right\}.$$

If f(t) = 1 t d dt g(t) and g(0) = 0, then it follows from (3.6.8) that

$$\bar{f}(s) = \mathscr{L}\left\{\frac{1}{t}\frac{d}{dt} g(t)\right\} = \int_s^\infty \mathscr{L}\left\{\frac{d}{dt} g(t)\right\} ds = \int_s^\infty x\, \bar{g}(x)\, dx.$$

(b) In view of uniform convergence of (3.2.5), the interchange of the order of integration is valid so that

$$\int_0^\infty \bar{f}(s)\, ds = \int_0^\infty ds \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty f(t) \left[\frac{e^{-st}}{-t}\right]_0^\infty dt = \int_0^\infty \frac{f(t)}{t}\, dt.$$

This proves the theorem.

For example,

$$\int_0^\infty \frac{\sin t}{t}\,dt = \int_0^\infty \mathscr{L}\{\sin t\}\,ds = \int_0^\infty \frac{ds}{s^2+1} = \left[\tan^{-1} s\right]_0^\infty = \frac{\pi}{2}.$$

Example 3.6.2 Show that

$$\text{(a)}\ \mathscr{L}\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left(\frac{a}{s}\right), \qquad \text{(b)}\ \mathscr{L}\left\{\frac{e^{-a^2/4t}}{\sqrt{\pi t^3}}\right\} = \frac{2}{a}\exp(-a\sqrt{s}),$$

$$\text{(c)}\ \mathscr{L}\left\{\frac{e^{bt}-e^{at}}{t}\right\} = \ln\left(\frac{s-a}{s-b}\right), \qquad \text{(d)}\ \mathscr{L}\left\{\frac{\cos bt - \sin at}{t}\right\} = \ln\sqrt{\frac{s^2+a^2}{s^2+b^2}}.$$

(a) Using (3.6.8), we obtain

$$\mathscr{L}\left\{\frac{\sin at}{t}\right\} = a\int_s^\infty \frac{ds}{s^2+a^2} = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{a}{s}\right).$$

$$\text{(b)}\ \mathscr{L}\left\{\frac{1}{t}\cdot\frac{e^{-a^2/4t}}{\sqrt{\pi t}}\right\} = \int_s^\infty \bar{f}(s)ds = \int_s^\infty \frac{e^{-a\sqrt{s}}}{\sqrt{s}}ds, \text{ by Table B-4 of Laplace}$$

Transforms, which is, by putting a $\sqrt{s} = x$,

(c) Using (3.6.8), we obtain

$$\mathscr{L}\left\{\frac{e^{bt}-e^{at}}{t}\right\} = \int_s^\infty \left[\frac{1}{s-b} - \frac{1}{s-a}\right\} ds$$

$$= \lim_{X\to\infty}\int_s^X \left\{\frac{1}{s-b} - \frac{1}{s-a}\right] ds$$

$$= \lim_{X\to\infty}\left[\ln\frac{X-b}{X-a} - \ln\frac{s-b}{s-a}\right]$$

$$= \ln(1) - \ln\frac{s-b}{s-a} = \ln\frac{s-a}{s-b}.$$

(d) In this case, we have

$$
\begin{aligned}
\mathscr{L}\left\{\frac{\cos bt-\cos at}{t}\right\} &= \int_s^\infty \mathscr{L}\left\{(\cos bt-\cos at)\right\} \\
&= \int_s^\infty \left\{\frac{s}{s^2+b^2}-\frac{s}{s^2+a^2}\right\} ds \\
&= \frac{1}{2}\lim_{X\to\infty}\int_s^X \left\{\frac{2s}{s^2+b^2}-\frac{2s}{s^2+a^2}\right\} ds \\
&= \frac{1}{2}\lim_{X\to\infty}\left[\ln\frac{X^2+b^2}{X^2+a^2}-\ln\frac{s^2+b^2}{s^2+a^2}\right] \\
&= \frac{1}{2}\lim_{X\to\infty}\left[\ln\frac{X^2+b^2}{X^2+a^2}\right]+\frac{1}{2}\ln\frac{s^2+a^2}{s^2+b^2}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathscr{L}\left\{\frac{\cos bt-\cos at}{t}\right\} &= \frac{1}{2}\ln 1+\ln\sqrt{\frac{s^2+a^2}{s^2+b^2}} \\
&= \ln\sqrt{\frac{s^2+a^2}{s^2+b^2}}.
\end{aligned}
$$

THEOREM 3.6.4 (The Laplace Transform of an Integral).

If L {f(t)} = ¯f(s), then

$$
\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{\bar{f}(s)}{s}. \qquad (3.6.10)
$$

PROOF We write

$$
g(t) = \int_0^t f(\tau)d\tau
$$

so that g(0) = 0 and g (t) = f(t). Then it follows from (3.4.12) that

$$
\bar{f}(s) = \mathscr{L}\{f(t)\} = \mathscr{L}\{g'(t)\} = s\,\bar{g}(s) = s\,\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\}.
$$

The result is obtained by multiplying both sides by s (3.6.10). We can see that the Laplace transform of an integral corresponds to the division of the transform of its integrand by s, which is a useful property to know. The result (3.6.10) may be used to determine the value of the inverse Laplace transform (3.6.11).

Example 3.6.3 Use result (3.6.10) to find

(a) $\mathscr{L}\left\{\int_0^t \tau^n e^{-a\tau} d\tau\right\}$, (b) $\mathscr{L}\{Si(at)\} = \mathscr{L}\left\{\int_0^t \frac{\sin a\tau}{\tau} d\tau\right\}$.

(c) $\mathscr{L}\left\{\int_0^t (f * g)(\tau) d\tau\right\}$

(a) We know

$$\mathscr{L}\{t^n e^{-at}\} = \frac{n!}{(s+a)^{n+1}}.$$

It follows from (3.6.10) that

$$\mathscr{L}\left\{\int_0^t \tau^n e^{-a\tau} d\tau\right\} = \frac{n!}{s(s+a)^{n+1}}.$$

(b) Using (3.6.10) and Example 3.6.2(a), we obtain

$$\mathscr{L}\left\{\int_0^t \frac{\sin a\tau}{\tau} d\tau\right\} = \frac{1}{s}\tan^{-1}\left(\frac{a}{s}\right).$$

(c) By Theorem 3.6.4, we have

$$\mathscr{L}\left\{\int_0^t (f * g)(\tau)\, d\tau\right\} = \frac{1}{s}\mathscr{L}\{(f * g)(t)\} = \frac{1}{s}\bar{f}(s)\bar{g}(s).$$

3.7 THE INVERSE OF LAPLACE TRANSFORM AND EXAMPLES

A prior demonstration proved that the Laplace transform f(s) of a given function f(t) may be determined by direct integration of the function f(t). We'll now have a look at the inverse of the issue. How can we determine the unknown function f(t) given a Laplace transform f(s) of the unknown function f(t)? This is primarily concerned with finding a solution to the integral equation (see below).

$$\int_0^\infty e^{-st} f(t)dt = \bar{f}(s). \qquad (3.7.1)$$

At this stage, it is rather difficult to handle the problem as it is. However, in simple cases, we can find the inverse transform from Table B-4 of Laplace transforms. For example

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1, \quad \mathscr{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos\ at.$$

For the most part, four approaches may be used to derive the inverse Laplace transform. These are: I partial fraction decomposition; (ii) the Convolution Theorem; (iii) contour integration of the Laplace Inversion Integral; and (iv) Heaviside's Expansion Theorem.

(i) Partial Fraction Decomposition Method

If

$$\bar{f}(s) = \frac{\bar{p}(s)}{\bar{q}(s)}, \qquad (3.7.2)$$

With respect to the case where p(s) and q(s) are polynomials in s and the degree of the polynomial p(s) is less than the degree of the polynomial q(s), it is possible to express f(s) as the sum of terms that can be inverted by using a table of Laplace transforms using the method of partial fractions. We demonstrate the approach with the help of simple illustrations.

Example 3.7.1 Show that

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s-a)}\right\} = \frac{1}{a}(e^{at}-1)$$

Where a is a constant. Using partial fraction, we obtain

$$\left\{\frac{1}{s(s-a)}\right\}=\frac{1}{a}\left\{\frac{1}{s-a}-\frac{1}{s}\right\}.$$

Thus, we have

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{s(s-a)}\right\}&=\mathscr{L}^{-1}\left[\frac{1}{a}\left\{\frac{1}{s-a}-\frac{1}{s}\right\}\right]\\&=\frac{1}{a}\left[\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\}-\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}\right]\\&=\frac{1}{a}(e^{at}-1).\end{aligned}$$

Example 3.7.2 Show that

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\}=\frac{1}{b^2-a^2}\left(\frac{\sin at}{a}-\frac{\sin bt}{b}\right).$$

We write

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\}&=\frac{1}{b^2-a^2}\left[\mathscr{L}^{-1}\left\{\frac{1}{s^2+a^2}-\frac{1}{s^2+b^2}\right\}\right]\\&=\frac{1}{(b^2-a^2)}\left(\frac{\sin at}{a}-\frac{\sin bt}{b}\right).\end{aligned}$$

Example 3.7.3 Find

$$\mathscr{L}^{-1}\left\{\frac{s+7}{s^2+2s+5}\right\}.$$

We have

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s+7}{(s+1)^2+4}\right\}&=\mathscr{L}^{-1}\left\{\frac{s+1+6}{(s+1)^2+2^2}\right\}\\&=\mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\}+3\mathscr{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}\\&=e^{-t}\cos 2t+3e^{-t}\sin 2t.\end{aligned}$$

Example 3.7.4 Evaluate the following inverse Laplace transform

$$\mathscr{L}^{-1}\left\{\frac{2s^2+5s+7}{(s-2)(s^2+4s+13)}\right\}.$$

We have

$$\mathscr{L}^{-1}\left\{\frac{2s^2+5s+7}{(s-2)(s^2+4s+13)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s-2}+\frac{s+2}{(s+2)^2+3^2}+\frac{1}{(s+2)^2+3^2}\right\}$$
$$= \mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\}$$
$$+\frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{(s+2)^2+3^2}\right\}$$
$$= e^{2t}+e^{-2t}\cos 3t+\frac{1}{3}e^{-2t}\sin 3t.$$

(ii) Convolution Theorem

We shall apply the convolution theorem for calculation of inverse Laplace transforms as follows :

Example 3.7.5

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s-a)}\right\} = 1 * e^{at} = \int_0^t e^{a\tau}\,d\tau = \frac{(e^{at}-1)}{a}.$$

Example 3.7.6 Show that

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} = \frac{1}{a^2}\left(t-\frac{1}{a}\sin at\right).$$

Here we obtain

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} = t * \frac{\sin at}{a}$$
$$= \frac{1}{a}\int_0^t (t-\tau)\sin a\tau\; d\tau$$
$$= \frac{t}{a}\int_0^t \sin a\tau\, d\tau - \frac{1}{a}\int_0^t \tau\sin a\tau\, d\tau$$
$$= \frac{1}{a^2}\left(t-\frac{1}{a}\sin at\right).$$

Example 3.7.7 Show that

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}=\frac{1}{2a^3}$$ (sin at − at cos at).

Using the Convolution Theorem 3.5.1, we obtain

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}&=\frac{\sin at}{a}*\frac{\sin at}{a}\\&=\frac{1}{a^2}\int_0^t \sin a\tau \sin a(t-\tau)d\tau\\&=\frac{1}{2a^3}(\sin at - at\cos at).\end{aligned}$$

Example 3.7.8 Show that

$$\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}(s-a)}\right\}=\frac{e^{at}}{\sqrt{a}}\,erf(\sqrt{at}).$$

Here we have

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}(s-a)}\right\}&=\frac{1}{\sqrt{\pi t}}*e^{at},\,(a>0)\\&=\frac{1}{\sqrt{\pi}}\int_0^t\frac{1}{\sqrt{\tau}}e^{a(t-\tau)}d\tau\\&=\frac{2\,e^{at}}{\sqrt{\pi a}}\int_0^{\sqrt{at}}e^{-x^2}dx,\quad(\text{putting }\sqrt{a\tau}=x)\\&=\frac{e^{at}}{\sqrt{a}}\,erf(\sqrt{at}).\end{aligned}\qquad(3.7.3)$$

Example 3.7.9 Show that

$$\mathscr{L}^{-1}\left\{\frac{1}{s}e^{-a\sqrt{s}}\right\}=erfc\left(\frac{a}{2\sqrt{t}}\right).\qquad(3.7.4)$$

In view of Example 3.6.2(b), and the Convolution Theorem 3.5.1, we obtain

$$\mathscr{L}^{-1}\left\{\frac{1}{s}e^{-a\sqrt{s}}\right\} = 1*\frac{a}{2}\frac{e^{-a^2/4t}}{\sqrt{\pi t^3}}$$
$$= \frac{a}{2\sqrt{\pi}}\int_0^t \frac{e^{-a^2/4\tau}}{\tau^{3/2}}d\tau,$$

Which is, by putting?

$$\frac{a}{2\sqrt{\tau}} = x$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s}e^{-a\sqrt{s}}\right\} = \frac{2}{\sqrt{\pi}}\int_{\frac{a}{2\sqrt{t}}}^{\infty} e^{-x^2}dx = erfc\left(\frac{a}{2\sqrt{t}}\right).$$

Example 3.7.10 Show that

$$\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}+a}\right\} = \frac{1}{\sqrt{\pi t}} - a\exp(ta^2)erfc(a\sqrt{t}). \qquad (3.7.5)$$

We have

$$\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}+a}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}} - \frac{a}{\sqrt{s}(\sqrt{s}+a)}\right\}$$
$$= \mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} - a\mathscr{L}^{-1}\left\{\frac{\sqrt{s}-a}{\sqrt{s}(s-a^2)}\right\}$$
$$= \mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} - a\mathscr{L}^{-1}\left\{\frac{1}{s-a^2}\right\} + a^2\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}(s-a^2)}\right\}$$
$$= \frac{1}{\sqrt{\pi t}} - a\exp(a^2t) + a\exp(a^2t)\,erf(a\sqrt{t}), \qquad \text{by (3.7.3)}$$
$$= \frac{1}{\sqrt{\pi t}} - a\exp(a^2t)erfc(a\sqrt{t}).$$

Example 3.7.11 If f(t) = L −1{ ¯f(s)}, then

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\bar{f}(s)\right\} = \int_0^t f(x)dx. \qquad (3.7.6)$$

We have, by the Convolution Theorem 3.5.1 with g(t) = 1 so that ¯g(s) = 1

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\bar{f}(s)\right\} = \int_0^t f(t-\tau)d\tau,$$
$$= \int_0^t f(x)\,dx \qquad \text{by putting} \quad t-\tau = x.$$

(iii) Contour Integration of the Laplace Inversion Integral

In Section 3.2, in inverse Laplace transform is defined by the complex integral formula

$$\mathscr{L}^{-1}\{\bar{f}(s)\} = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}(s)ds, \qquad (3.7.7)$$

Where c is a suitable real constant and f(s) is an analytic function of the complex variable s in the right half-plane, c is a suitable real constant and f(s) is an analytic function of the complex variable s s>a s>a s>a s>a s>a s>a s>a s>a s>a s>a s>a s>a It is dependent on how the singularities of f behave in order to determine how to evaluate (3.7.7) in detail (s). Typically, f(s) is a single-valued function with a finite or enumerably infinite number of polar singularities, as opposed to a multivalued function. It is common for it to have branch points. The path of integration is the straight line L (see Figure 3.4(a)) in the complex s-plane with the equation s = c + iR, iR, Re s = c being chosen so that all of the singularities of the integrand of (3.7.7) lie to the left of the line L. The path of integration is the straight line L (see Figure 3.4(a)) in the complex s-plane with the equation s = c + The Bromwich Contour is the name given to this line. Figure 3.4(a) illustrates how the Bromwich Contour is closed by an arc of a circle of radius R, and then the limit as R is used to extend the contour of integration to infinity, ensuring that all of the singularities of f(s) are included inside the contour of integration. When f(s) has a branch point at the origin, we construct the modified contour of integration by cutting along the negative real axis and drawing a tiny semicircle around the origin, as illustrated in Figure 3.4. Figure 3.4: Modified contour of integration with branch point at the origin (b).

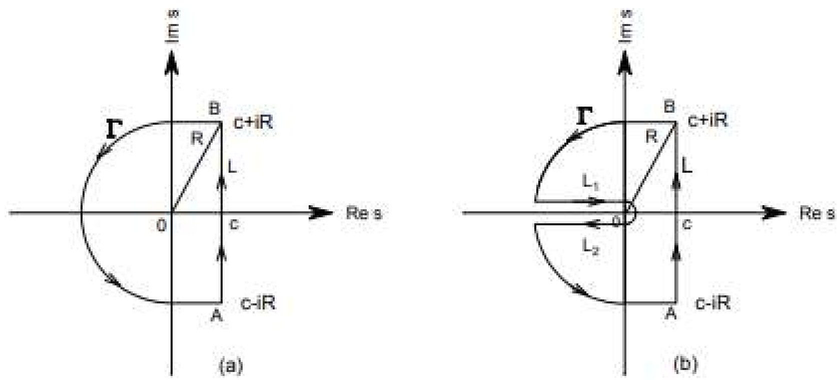


Figure 3.4 the Bromwich contour and the contour of integration

In either case, the Cauchy Residue Theorem is used to evaluate the integral

$$\int_L e^{st}\bar{f}(s)ds + \int_\Gamma e^{st}\bar{f}(s)ds = \int_C e^{st}\bar{f}(s)ds$$

= 2πi × [sum of the residues of est ¯f(s) at the poles inside C]. (3.7.8)

Letting R → ∞, the integral over Γ tends to zero, and this is true in most problems of interest. Consequently, result (3.7.7) reduces to the form

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{c-iR}^{c+iR} e^{st}\bar{f}(s)ds = \text{sum of the residues of } e^{st}\bar{f}(s) \text{ at the poles of } \bar{f}(s). \quad (3.7.9)$$

We illustrate the above method of evaluation by simple examples.

Example 3.7.12

If $\bar{f}(s) = \frac{s}{s^2+a^2}$, show that

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}(s)ds = \cos at.$$

Clearly, the integrand has two simple poles at s = ±ia and the residues at these poles are

$$R_1 = \text{Residue of } e^{st}\bar{f}(s) \text{ at } s = ia$$
$$= \lim_{s\to ia}(s-ia)\frac{s\,e^{st}}{(s^2+a^2)} = \frac{1}{2}e^{iat}.$$
$$R_2 = \text{Residue of } e^{st}\bar{f}(s) \text{ at } s = -ia$$
$$= \lim_{s\to -ia}(s+ia)\frac{s\,e^{st}}{(s^2+a^2)} = \frac{1}{2}e^{-iat}.$$

Hence,

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}(s)ds = R_1 + R_2 = \frac{1}{2}(e^{iat}+e^{-iat}) = \cos at,$$

As obtained earlier.

If ¯g(s) = est ¯f(s) has a pole of order n at s = z, then the residue R1 of ¯g(s) at this pole is given by the formula

$$R_1 = \lim_{s\to z}\frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}[(s-z)^n\bar{g}(s)]. \qquad (3.7.10)$$

If ¯g(s) = est ¯f(s) has a pole of order n at s = z, then the residue R1 of ¯g(s) at this pole is given by the formula

$$R_1 = \lim_{s\to z}\frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}[(s-z)^n\bar{g}(s)]. \qquad (3.7.10)$$

This is obviously true for a simple pole (n = 1) and for a double pole (n = 2).

Example 3.7.13 Evaluate

$$\mathscr{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}.$$

Clearly

$$\bar{g}(s) = e^{st}\bar{f}(s) = \frac{s\,e^{st}}{(s^2+a^2)^2}$$

Has double poles at s = ±ia. The residue formula (3.7.10) for double poles gives

$$R_1 = \lim_{s\to ia} \frac{d}{ds}\left[(s-ia)^2 \frac{s\,e^{st}}{(s^2+a^2)^2}\right]$$
$$= \lim_{s\to ia} \frac{d}{ds}\left[\frac{s\,e^{st}}{(s+ia)^2}\right] = \frac{t\,e^{iat}}{4ia}.$$

Similarly, the residue at the double pole at s = −ia is (−t e−iat)/4ia.

Thus,

$$f(t) = \text{Sum of the residues} = \frac{t}{4ia}(e^{iat} - e^{-iat}) = \frac{t}{2a}\sin at, \qquad (3.7.11)$$

as given in Table B-4 of Laplace transforms.

Example 3.7.14 Evaluate

$$\mathscr{L}^{-1}\left\{\frac{\cosh(\alpha x)}{s\cosh(\alpha \ell)}\right\}, \qquad \alpha = \sqrt{\frac{s}{a}}.$$

We have

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\,\frac{\cosh(\alpha x)}{\cosh(\alpha \ell)}\,\frac{ds}{s}.$$

Clearly, the integrand has simple poles at s = 0 and s = sn $= -(2n+1)^2\frac{a\pi^2}{4\ell^2}$, where n = 0,1,2, ...

R1 = Residue at the pole s = 0 is 1, and Rn = Residue at the pole s = sn is

$$\frac{\exp(-s_n t)\cosh\left\{i(2n+1)\frac{\pi x}{2\ell}\right\}}{\left[s\frac{d}{ds}\left\{\cosh l\sqrt{\frac{s}{a}}\right\}\right]_{s=s_n}}$$
$$= \frac{4(-1)^{n+1}}{(2n+1)\pi}\exp\left[-\left\{\frac{(2n+1)\pi}{2\ell}\right\}^2 at\right]\cos\left\{(2n+1)\frac{\pi x}{2\ell}\right\}.$$

Thus

$$f(t) = \text{Sum of the residues at the poles}$$
$$= 1 + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)}\exp\left[-(2n+1)^2\frac{\pi^2 at}{4\ell^2}\right]$$
$$\times\cos\left\{(2n+1)\frac{\pi x}{2\ell}\right\}, \qquad (3.7.12)$$

as given later by the Heaviside Expansion Theorem 3.7.1.

Example 3.7.15 Show that

$$f(t) = \mathscr{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{s}\exp(st - a\sqrt{s})ds$$
$$= erfc\left(\frac{a}{2\sqrt{t}}\right). \qquad (3.7.13)$$

The integrand has a branch point at the value of s = 0. We use the contour of integration shown in Figure 3.4(b), which removes the branch point at s = 0 and therefore simplifies the problem. Consequently, the Cauchy Fundamental Theorem provides

$$\frac{1}{2\pi i}\left[\int_L + \int_\Gamma + \int_{L_1} + \int_{L_2} + \int_\gamma\right]\exp(st - a\sqrt{s})\,\frac{ds}{s} = 0. \qquad (3.7.14)$$

It is shown that the integral on Γ tends to zero as R → ∞, and that on L gives the Bromwich integral. We now evaluate the remaining three integrals in (3.7.14). On L1, we have s = reiπ = −r and

$$\int_{L_1} \exp(st - a\sqrt{s})\,\frac{ds}{s} = \int_{-\infty}^{0} \exp(st - a\sqrt{s})\,\frac{ds}{s} = -\int_{0}^{\infty} \exp\{-(rt + ia\sqrt{r})\}\,\frac{dr}{r}.$$

On L2, s = re−iπ = −r and

$$\int_{L_2} \exp(st - a\sqrt{s})\,\frac{ds}{s} = \int_{0}^{-\infty} \exp(st - a\sqrt{s})\,\frac{ds}{s} = \int_{0}^{\infty} \exp\{-rt + ia\sqrt{r}\}\,\frac{dr}{r}.$$

Thus, the integrals along L1 and L2 combined yield

$$-2i\int_{0}^{\infty} e^{-rt}\sin(a\sqrt{r})\,\frac{dr}{r} = -4i\int_{0}^{-\infty} e^{-x^2 t}\,\frac{\sin ax}{x}dx, \qquad (\sqrt{r} = x). \qquad (3.7.15)$$

Integrating the following standard integral with respect to β

$$\int_{0}^{\infty} e^{-x^2\alpha^2}\cos(2\beta x)dx = \frac{\sqrt{\pi}}{2\alpha}\exp\left(-\frac{\beta^2}{\alpha^2}\right), \qquad (3.7.16)$$

We obtain

$$\begin{aligned}\frac{1}{2}\int_{0}^{\infty} e^{-x^2\alpha^2}\,\frac{\sin 2\beta x}{x}dx &= \frac{\sqrt{\pi}}{2\alpha}\int_{0}^{\beta}\exp\left(-\frac{\beta^2}{\alpha^2}\right)d\beta \\ &= \frac{\sqrt{\pi}}{2}\int_{0}^{\beta/\alpha} e^{-u^2}\,du, \qquad (\beta = \alpha u) \\ &= \frac{\pi}{4}\,erf\left(\frac{\beta}{\alpha}\right). \qquad (3.7.17)\end{aligned}$$

In view of (3.7.17), result (3.7.15) becomes

$$-4i\int_0^\infty \exp(-tx^2)\,\frac{\sin ax}{x}\,dx = -2\pi i\, erf\left(\frac{a}{2\sqrt{t}}\right). \qquad (3.7.18)$$

Finally, on γ, we have s = reiθ, ds = ireiθdθ, and

$$\int_\gamma |\exp(st - a\sqrt{s})|\frac{ds}{s} = i\int_\pi^{-\pi} \exp\left(rt\cos\theta - a\sqrt{r}\cos\frac{\theta}{2}\right)d\theta$$
$$= i\int_{-\pi}^{\pi} d\theta = 2\pi i, \qquad (3.7.19)$$

In this case, the limit as r 0 is employed, and the integration from r to r is swapped to produce r in the counterclockwise manner, resulting in r. In this way, the final result is derived from (3.7.14), (3.7.18), and (3.7.19) in the following form:

$$\mathscr{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(st - a\sqrt{s})\,\frac{ds}{s}$$
$$= \left[1 - erf\left(\frac{a}{2\sqrt{t}}\right)\right] = erfc\left(\frac{a}{2\sqrt{t}}\right).$$

(iv) Heaviside’s Expansion Theorem

Suppose ¯f(s) is the Laplace transform of f(t), which has a Maclaurin power series expansion in the form

$$f(t) = \sum_{r=0}^{\infty} a_r\,\frac{t^r}{r!}. \qquad (3.7.20)$$

Taking the Laplace transform, it is possible to write formally

$$\bar{f}(s) = \sum_{r=0}^{\infty} \frac{a_r}{s^{r+1}}. \qquad (3.7.21)$$

Conversely, we can derive (3.7.20) from a given expansion (3.7.21). This kind of expansion is useful for determining the behavior of the solution for small time. Further, it provides an alternating way to prove the Tauberian theorems.

THEOREM 3.7.1 (Heaviside's Expansion Theorem)

If $\bar{f}(s)=\frac{\bar{p}(s)}{\bar{q}(s)}$, where ¯p(s) and ¯q(s) are polynomials in s and the degree of ¯q(s) is higher than that of ¯p(s), then

$$\mathscr{L}^{-1}\left\{\frac{\bar{p}(s)}{\bar{q}(s)}\right\}=\sum_{k=1}^{n}\frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)}\exp(t\alpha_k). \qquad (3.7.22)$$

Where αk are the distinct roots of the equation ¯q(s) = 0.

PROOF Without loss of generality, we can assume that the leading coefficient of ¯q(s) is unity and write distinct factors of ¯q(s) so that

$$\bar{q}(s)=(s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_k)\cdots(s-\alpha_n). \qquad (3.7.23)$$

Using the rules of partial fraction decomposition, we can write

$$\bar{f}(s)=\frac{\bar{p}(s)}{\bar{q}(s)}=\sum_{k=1}^{n}\frac{A_k}{(s-\alpha_k)}, \qquad (3.7.24)$$

Where Ak are arbitrary constants to be determined. In view of (3.7.23), we find

$$\bar{p}(s)=\sum_{k=1}^{n}A_k(s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_{k-1})(s-\alpha_{k+1})\cdots(s-\alpha_n).$$

Substitution of s = αk gives

$$\bar{p}(\alpha_k)=A_k(\alpha_k-\alpha_1)(\alpha_k-\alpha_2)\cdots(\alpha_k-\alpha_{k+1})\cdots(\alpha_k-\alpha_n), \qquad (3.7.25)$$

Where k = 1, 2, 3,...,n.

Differentiation of (3.7.23) yields

$$\bar{q}'(s) = \sum_{k=1}^{n} (s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_{k-1})(s-\alpha_{k+1})\cdots(s-\alpha_n).$$

Whence it follows that

$$\bar{q}'(\alpha_k) = (\alpha_k-\alpha_1)(\alpha_k-\alpha_2)\cdots(\alpha_k-\alpha_{k-1})(\alpha_k-\alpha_{k+1})\cdots(\alpha_k-\alpha_n). \tag{3.7.26}$$

From (3.7.25) and (3.7.26), we find

$$A_k = \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)},$$

And hence,

$$\frac{\bar{p}(s)}{\bar{q}(s)} = \sum_{k=1}^{n} \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)} \frac{1}{(s-\alpha_k)}. \tag{3.7.27}$$

Inversion gives immediately

$$\mathscr{L}^{-1}\left\{\frac{\bar{p}(s)}{\bar{q}(s)}\right\} = \sum_{k=1}^{n} \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)} \exp(t\alpha_k).$$

This proves the theorem. We give some examples of this theorem.

Example 3.7.16 We consider

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2-3s+2}\right\}.$$

Here $\bar{p}(s) = s$, and $\bar{q}(s) = s^2 - 3s + 2 = (s-1)(s-2)$. Hence,

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2-3s+2}\right\} = \frac{\bar{p}(2)}{\bar{q}'(2)} e^{2t} + \frac{\bar{p}(1)}{\bar{q}'(1)} e^{t} = 2\,e^{2t} - e^{t}.$$

Example 3.7.17 Use Heaviside's power series expansion to evaluate

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\frac{\sinh x\sqrt{s}}{\sinh\sqrt{s}}\right\}, \qquad 0<x<1, \qquad s>0.$$

We have

$$\begin{aligned}
\frac{1}{s}\frac{\sinh x\sqrt{s}}{\sinh\sqrt{s}} &= \frac{1}{s}\left(\frac{e^{x\sqrt{s}}-e^{-x\sqrt{s}}}{e^{\sqrt{s}}-e^{-\sqrt{s}}}\right)\\
&= \frac{1}{s}\frac{e^{-(1-x)\sqrt{s}}-e^{-(1+x)\sqrt{s}}}{1-e^{-2\sqrt{s}}}\\
&= \frac{1}{s}\left[e^{-(1-x)\sqrt{s}}-e^{-(1+x)\sqrt{s}}\right]\left(1-e^{-2\sqrt{s}}\right)^{-1}\\
&= \frac{1}{s}\left[e^{-(1-x)\sqrt{s}}-e^{-(1+x)\sqrt{s}}\right]\sum_{n=0}^{\infty}\exp(-2n\sqrt{s})\\
&= \frac{1}{s}\sum_{n=0}^{\infty}\left[\exp\{-(1-x+2n)\sqrt{s}\}-\exp\{-(1+x+2n)\sqrt{s}\}\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathscr{L}^{-1} &\left\{\frac{1}{s}\frac{\sinh x\sqrt{s}}{\sinh\sqrt{s}}\right\}\\
&= \mathscr{L}^{-1}\left\{\frac{1}{s}\sum_{n=0}^{\infty}\left[\exp\{-(1-x+2n)\sqrt{s}\}-\exp\{-(1+x+2n)\sqrt{s}\}\right]\right\}\\
&= \sum_{n=0}^{\infty}\left[\mathit{erfc}\left(\frac{1-x+2n}{2\sqrt{t}}\right)-\mathit{erfc}\left(\frac{1+x+2n}{2\sqrt{t}}\right)\right].
\end{aligned}$$

Example 3.7.18

If $\alpha = \sqrt{\frac{s}{a}}$, show that

$$\mathscr{L}^{-1}\left[\frac{\cosh\alpha x}{s\cosh\alpha\ell}\right] = 1-\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k\cos\left\{\left(k+\frac{1}{2}\right)\frac{\pi x}{\ell}\right\}\exp\left[-(2k+1)^2\frac{a\pi^2 t}{4\ell^2}\right]}{(2k+1)}. \tag{3.7.28}$$

In this case, we write

$$\mathscr{L}^{-1}\{\bar{f}(s)\} = \mathscr{L}^{-1}\left\{\frac{\bar{p}(s)}{\bar{q}(s)}\right\} = \mathscr{L}^{-1}\left\{\frac{\cosh\alpha x}{s\cosh\alpha\ell}\right\}.$$

Clearly, the zeros of ¯f(s) are at s = 0 and at the roots of cosh α= 0, that is,

$$\text{at } s = s_k = a\left(k+\frac{1}{2}\right)^2\left(\frac{\pi i}{\ell}\right)^2, \quad k = 0, 1, 2, \ldots. \text{ Thus,}$$

$$\alpha_k = \sqrt{\frac{s_k}{a}} = \left(k+\frac{1}{2}\right)\frac{\pi i}{\ell}, \qquad k = 0, 1, 2, \ldots.$$

Here ¯p(s) = cosh(αx), ¯q(s) = s cosh(α). In order to apply the Heaviside Expansion Theorem, we need

$$\bar{q}'(s) = \frac{d}{ds}(s\cosh\alpha\ell) = \cosh(\alpha\ell) + \frac{1}{2}\alpha\ell\sinh(\alpha\ell).$$

For the zero $s = 0,\ \bar{q}'(0) = 1,$ and for the zeros at s = sk,

$$\begin{aligned}\bar{q}'(s_k) &= \frac{1}{2}\left(k+\frac{1}{2}\right)\pi i \cdot \sinh\left[\left(k+\frac{1}{2}\right)\pi i\right]\\ &= (2k+1)\frac{\pi i}{4}\cdot i\sin\left[\left(k+\frac{1}{2}\right)\pi\right]\\ &= -(2k+1)\frac{\pi}{4}\cdot\cos k\pi = (-1)^{k+1}(2k+1)\frac{\pi}{4}.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{\cosh\alpha x}{s\cosh\alpha\ell}\right\} &= 1 + \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)}\cosh\left[(2k+1)\frac{\pi i x}{2\ell}\right]\exp(ts_k)\\ &= 1 - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{(2k+1)}\cos\left[(2k+1)\frac{\pi x}{2\ell}\right]\\ &\qquad\times\exp\left[-\left(k+\frac{1}{2}\right)^2\frac{\pi^2 a t}{\ell^2}\right].\end{aligned}$$

CHAPTER 4

APPLICATIONS OF LAPLACE TRANSFORMS

In addition to receiving particular attention since antiquity, mathematical sciences are now gaining even greater interest now as a result of their effect on business and the arts. The convergence of theory and practise produces the most advantageous outcomes, and it is not just the practical side that benefits; research advances as a consequence of its effect, as it finds new areas of study and new elements of mathematical sciences..."

"... Partial differential equations serve as the foundation for all physical theorems...." In the theory of sound in gases, liquids, and solids, in the examination of elasticity, and in optics, partial differential equations create fundamental principles of nature that may be tested against experimental evidence."

4.1 INTRODUCTION

Ordinary or partial differential equations with proper beginning or boundary conditions may be used to explain a wide range of physical issues of interest. In applied and engineering sciences, these issues are often stated as initial value problems, boundary value problems, or initial-boundary value problems, which seem to be mathematically more rigorous and physically realistic than other types of problems. When it comes to finding answers to these difficulties, the Laplace transform approach is extremely effective. For the solution of the response of a linear system controlled by an ordinary differential equation to the starting data and/or to an external disturbance, the approach is quite successful (or external input function). For more specificity, we look for the solution of a linear system for its state at a future time after $t = 0$ that is caused by the initial state at $t = 0$ and/or the disturbance delivered at $t > 0$. Throughout this chapter, we will discuss the solutions of ordinary and partial differential equations that occur in the fields of mathematics, physical science, and engineering. This chapter also covers the applications of Laplace transforms to the solutions of certain integral equations and boundary value issues, as well as the applications of Laplace transforms to the solution of certain boundary value problems. There are many cases in which it is shown that the Laplace transform may also be utilised efficiently for the evaluation of certain definite integrals. A few examples of solutions of difference and differential equations utilising the Laplace transform method are also included in this section. Solving a number of initial-boundary value issues using the combined Laplace and Fourier transforms demonstrates how to make efficient use of this technique. With the help of examples, the application of Laplace transforms to the issue of summation of infinite series in closed form is shown. At the conclusion of this chapter, it should be highlighted that the examples provided in this chapter are simply typical of the large range of issues that may be handled by using the Laplace transform approach.

4.2 SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

A linear system controlled by a differential equation may be analysed in terms of its fundamental properties in reaction to starting data and/or to an external disturbance, as explained in the introduction to this chapter. The examples that follow demonstrate the use of the Laplace transform in the solution of some initial value problems that are represented by ordinary differential equations.

Example 4.2.1 (Initial Value Problem)

We consider the first-order ordinary differential equation

$$\frac{dx}{dt} + px = f(t), \qquad t > 0, \qquad (4.2.1)$$

With the initial condition

$$x(t=0) = a, \qquad (4.2.2)$$

Where p and a are constants and f(t) is an external input function so that its Laplace transform exists.

Application of the Laplace transform $\bar{x}(s)$ of the function x(t) gives

$$s\,\bar{x}(s) - x(0) + p\,\bar{x}(s) = \bar{f}(s),$$

Or

$$\bar{x}(s) = \frac{a}{s+p} + \frac{\bar{f}(s)}{s+p}. \qquad (4.2.3)$$

The inverse Laplace transform together with the Convolution Theorem leads to the solution

$$x(t) = ae^{-pt} + \int_0^t f(t-\tau)e^{-p\tau}d\tau. \qquad (4.2.4)$$

This naturally results in two terms in the solution: the first term, which corresponds to the response of the starting condition, and the second term which is fully attributable to the external input function f (which is the result of the external input function f) (t).

In particular, if f(t) = q = constant, then the solution (4.2.4) becomes

$$x(t) = \frac{q}{p} + \left(a - \frac{q}{p}\right)e^{-pt}. \qquad (4.2.5)$$

It is customary to refer to the steady-state solution as the first term of this equation since it is independent of time t. The second component, which is dependent on time t, is referred to as the transient solution. if $p > 0$, the transient solution decays to zero in the limit as t 0 and the steady-state solution is reached in the limit. When p is less than zero, on the other hand, the transient solution expands exponentially as time progresses, and the solution becomes unstable. Equation (4.2.1) provides the rule of natural growth or decay process with an external forcing function f(t) according to the condition $p > 0$ or p 0 in the context of a natural growth or decay process. It should be noted that when f(t) is equal to zero and p is greater than zero, the following equation (4.2.1) occurs quite often in chemical kinetics. The rate of chemical reactions may be described by such an equation.

Example 4.2.2 (Second-Order Ordinary Differential Equation

The second-order linear ordinary differential equation has the general form

$$\frac{d^2x}{dt^2} + 2p\frac{dx}{dt} + qx = f(t), \qquad t > 0. \qquad (4.2.6)$$

The initial conditions are

$$x(t) = a, \quad \frac{dx}{dt} = \dot{x}(t) = b \qquad \text{at } t = 0, \qquad (4.2.7\text{ab})$$

Where p, q, a and b are constants.

Application of the Laplace transform to this general initial value problem gives

$$s^2 \bar{x}(s) - s\,x(0) - \dot{x}(0) + 2p\{s\,\bar{x}(s) - x(0)\} + q\bar{x}(s) = \bar{f}(s).$$

The use of (4.2.7ab) leads to the solution for ¯x(s) as

$$\bar{x}(s) = \frac{(s+p)a + (b+pa) + \bar{f}(s)}{(s+p)^2 + n^2}, \qquad n^2 = q - p^2. \qquad (4.2.8)$$

The inverse transform gives the solution in three distinct forms depending on q > = < p2, and they are

$$x(t) = ae^{-pt}\cos nt + \frac{1}{n}(b+pa)e^{-pt}\sin nt + \frac{1}{n}\int_0^t f(t-\tau)e^{-p\tau}\sin n\tau d\tau, \qquad \text{when } n^2 = q - p^2 > 0, \quad (4.2.9)$$

$$x(t) = ae^{-pt} + (b+pa)\,t\,e^{-pt} + \int_0^t f(t-\tau)\,\tau\,e^{-p\tau}d\tau, \qquad \text{when } n^2 = q - p^2 = 0, \qquad (4.2.10)$$

$$x(t) = ae^{-pt}\cosh mt + \frac{1}{m}(b+pa)\,e^{-pt}\sinh mt + \frac{1}{m}\int_0^t f(t-\tau)\,e^{-p\tau}\sinh m\tau d\tau, \quad \text{when } m^2 = p^2 - q > 0. \, (4.2.11)$$

Example 4.2.3 (Higher-Order Ordinary Differential Equations) We solve the linear equation of order n with constant coefficients as

$$f(D)\{x(t)\} \equiv D^n x + a_1 D^{n-1}x + a_2 D^{n-2}x + \cdots + a_n x = \phi(t), \quad t > 0, \qquad (4.2.12)$$

With the initial conditions

$$x(t)=x_0, \quad Dx(t)=x_1, \quad D^2x(t)=x_2,\dots,D^{n-1}x(t)=x_{n-1}, \quad \text{at } t=0, \tag{4.2.13}$$

Where $D=\frac{d}{dt}$ is the differential operator and x0, x1,...,xn−1 are constants. We take the Laplace transform of (4.2.12) to get

$$\begin{aligned}&(s^n\,\bar{x} - s^{n-1}\,x_0 - s^{n-2}\,x_1 - \cdots - s\,x_{n-2} - x_{n-1})\\ &\quad +a_1\left(s^{n-1}\,\bar{x} - s^{n-2}\,x_0 - s^{n-3}\,x_1 - \cdots - x_{n-2}\right)\\ &\quad +a_2\left(s^{n-2}\,\bar{x} - s^{n-3}\,x_0 - \cdots - x_{n-3}\right)\\ &\quad +\cdots + a_{n-1}(s\,\bar{x} - x_0) + a_n\,\bar{x} = \bar{\phi}(s).\end{aligned} \tag{4.2.14}$$

Or

$$\begin{aligned}&(s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n)\,\bar{x}(s)\\ &\quad = \bar{\phi}(s) + (s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-1})x_0\\ &\qquad + (s^{n-2} + a_1 s^{n-3} + \cdots + a_{n-2})x_1 + \cdots + (s+a_1)x_{n-2} + x_{n-1}\\ &\quad = \bar{\phi}(s) + \bar{\psi}(s),\end{aligned} \tag{4.2.15}$$

Where ψ¯(s) is made up of all terms on the right-hand side of (4.2.15) except φ¯(s), and is a polynomial in s of degree (n − 1).

Hence

$$\bar{f}(s)\,\bar{x}(s) = \bar{\phi}(s) + \bar{\psi}(s),$$

Where

$$\bar{f}(s) = s^n + a_1 s^{n-1} + \cdots + a_n.$$

Thus, the Laplace transform solution, ¯x(s) is

$$\bar{x}(s) = \frac{\bar{\phi}(s) + \bar{\psi}(s)}{\bar{f}(s)}. \qquad (4.2.16)$$

Inversion of (4.2.16) yields

$$x(t) = \mathscr{L}^{-1}\left\{\frac{\bar{\phi}(s)}{\bar{f}(s)}\right\} + \mathscr{L}^{-1}\left\{\frac{\bar{\psi}(s)}{\bar{f}(s)}\right\}. \qquad (4.2.17)$$

The inverse operation on the right can be carried out by partial fraction decomposition, by the Heaviside Expansion Theorem, or by contour integration.

Example 4.2.4 (Third-Order Ordinary Differential Equations) We solve

$$(D^3 + D^2 - 6D)x(t) = 0, \qquad D \equiv \frac{d}{dt}, \qquad t > 0, \qquad (4.2.18)$$

With the initial data

$$x(0) = 1, \ \dot{x}(0) = 0, \ \text{and } \ddot{x}(0) = 5. \qquad (4.2.19)$$

The Laplace transform of equation (4.2.18) gives

$$[s^3\bar{x} - s^2x(0) - s\,\dot{x}(0) - \ddot{x}(0)] + [s^2\bar{x} - s\,x(0) - \dot{x}(0)] - 6[s\,\bar{x} - x(0)] = 0.$$

In view of the initial conditions, we find

$$\bar{x}(s) = \frac{s^2 + s - 1}{s(s^2 + s - 6)} = \frac{s^2 + s - 1}{s(s+3)(s-2)}.$$

Or,

$$\bar{x}(s) = \frac{1}{6}\cdot\frac{1}{s} + \frac{1}{3}\cdot\frac{1}{s+3} + \frac{1}{2}\cdot\frac{1}{s-2}.$$

Inverting gives the solution

$$x(t) = \frac{1}{6} + \frac{1}{3}e^{-3t} + \frac{1}{2}e^{2t}. \qquad (4.2.20)$$

Example 4.2.5 (System of First-Order Ordinary Differential Equations Consider the system

$$\left.\begin{aligned}\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + b_1(t)\\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + b_2(t)\end{aligned}\right\} \qquad (4.2.21ab)$$

With the initial data

$$x_1(0) = x_{10} \quad \text{and} \quad x_2(0) = x_{20}, \qquad (4.2.22ab)$$

Where a11, a12, a21, a22 are constants.

Introducing the matrices

$$x \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \frac{dx}{dt} \equiv \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix}, \qquad A \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$
$$b(t) \equiv \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} \quad \text{and} \quad x_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix},$$

We can write the above system in a matrix differential system as

$$\frac{dx}{dt} = A\,x + b(t), \qquad x(0) = x_0. \qquad (4.2.23ab)$$

We take the Laplace transform of the system with the initial conditions to get

$$\begin{aligned}(s - a_{11})\bar{x}_1 - a_{12}\bar{x}_2 &= x_{10} + \bar{b}_1(s),\\ -a_{21}\bar{x}_1 + (s - a_{22})\bar{x}_2 &= x_{20} + \bar{b}_2(s).\end{aligned}$$

The solutions of this algebraic system are

$$\bar{x}_1(s) = \frac{\begin{vmatrix} x_{10} + \bar{b}_1(s) & -a_{12} \\ x_{20} + \bar{b}_2(s) & s - a_{22} \end{vmatrix}}{\begin{vmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{vmatrix}}, \quad \bar{x}_2(s) = \frac{\begin{vmatrix} s - a_{11} & x_{10} + \bar{b}_1(s) \\ -a_{21} & x_{20} + \bar{b}_2(s) \end{vmatrix}}{\begin{vmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{vmatrix}}. \qquad (4.2.24ab)$$

Expanding these determinants, results for ¯x1(s) and ¯x2(s) can readily be inverted, and the solutions for x1(t) and x2(t) can be found in closed forms.

Example 4.2.6 Solve the matrix differential system

$$\frac{dx}{dt} = Ax, \qquad x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad (4.2.25)$$

Where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}.$$

This system is equivalent to

$$\frac{dx_1}{dt} - x_2 = 0,$$
$$\frac{dx_2}{dt} + 2x_1 - 3x_2 = 0,$$

With

$$x_1(0) = 0 \quad \text{and} \quad x_2(0) = 1.$$

Taking the Laplace transform of the coupled system with the given initial data, we find

$$s\,\bar{x}_1 - \bar{x}_2 = 0,$$
$$2\bar{x}_1 + (s-3)\bar{x}_2 = 1.$$

This system has the solutions

$$\bar{x}_1(s) = \frac{1}{s^2 - 3s + 2} = \frac{1}{s-2} - \frac{1}{s-1},$$
$$\bar{x}_2(s) = \frac{s}{s^2 - 3s + 2} = \frac{2}{s-2} - \frac{1}{s-1}.$$

Inverting these results, we obtain

$$x_1(t) = e^{2t} - e^t, \qquad x_2(t) = 2\,e^{2t} - e^t.$$

In matrix notation, the solution is

$$x(t) = \begin{pmatrix} e^{2t} - e^t \\ 2\,e^{2t} - e^t \end{pmatrix}. \qquad (4.2.26)$$

Example 4.2.7 (Second-Order Coupled Differential System)

Solve the system

$$\left.\begin{aligned} \frac{d^2x_1}{dt^2} - 3x_1 - 4x_2 = 0 \\ \frac{d^2x_2}{dt^2} + x_1 + x_2 = 0 \end{aligned}\right\} \quad t > 0, \qquad (4.2.27)$$

With the initial conditions

$$x_1(t) = x_2(t) = 0; \qquad \frac{dx_1}{dt} = 2 \quad \text{and} \quad \frac{dx_2}{dt} = 0 \text{ at } t = 0. \qquad (4.2.28)$$

The use of the Laplace transform to (4.2.27) with (4.2.28) gives

$$\begin{aligned} (s^2 - 3)\bar{x}_1 - 4\bar{x}_2 = 2 \\ \bar{x}_1 + (s^2 + 1)\bar{x}_2 = 0. \end{aligned}$$

Then

$$\bar{x}_1(s) = \frac{2(s^2+1)}{(s^2-1)^2} = \frac{(s+1)^2 + (s-1)^2}{(s^2-1)^2} = \frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}.$$

Hence, the inversion yields

$$x_1(t) = t(e^t + e^{-t}). \qquad (4.2.29)$$

$$\bar{x}_2(s) = \frac{-2}{(s^2-1)^2} = \frac{1}{2}\left[\frac{1}{s-1} - \frac{1}{s+1} - \frac{1}{(s-1)^2} - \frac{1}{(s+1)^2}\right],$$

Which can be readily inverted to find?

$$x_2(t) = \frac{1}{2}(e^t - e^{-t} - t\,e^t - t\,e^{-t}). \qquad (4.2.30)$$

As an example, consider 4.2.8. (The Harmonic Oscillator in a Non-Resisting Medium). This is the differential equation of the oscillator in the presence of an external driving force, denoted by the letter f(t).

$$\frac{d^2x}{dt^2} + \omega^2 x = F\,f(t), \qquad (4.2.31)$$

Where ω is the frequency and F is a constant. The initial conditions are

$$x(t) = a, \quad \dot{x}(t) = U \qquad \text{at } t = 0, \qquad (4.2.32)$$

Where a and U are constants.

Taking the Laplace transform of (4.2.31) with the initial conditions, we obtain

$$(s^2 + \omega^2)\bar{x}(s) = sa + U + F\,\bar{f}(s).$$

Or,

$$\bar{x}(s) = \frac{as}{s^2+\omega^2} + \frac{U}{s^2+\omega^2} + \frac{F\,\bar{f}(s)}{s^2+\omega^2}. \qquad (4.2.33)$$

Inversion together with the convolution theorem yields

$$x(t) = a\cos\omega t + \frac{U}{\omega}\sin\omega t + \frac{F}{\omega}\int_0^t f(t-\tau)\sin\omega\tau d\tau \qquad (4.2.34)$$

$$= A\ \cos(\omega t - \phi) + \frac{F}{\omega}\int_0^t f(t-\tau)\ \sin\omega\tau d\tau, \qquad (4.2.35)$$

where $A = \left(a^2 + \frac{U^2}{\omega^2}\right)^{1/2}$ and $\phi = \tan^{-1}\left(\frac{U}{\omega a}\right)$.

The answer (4.2.35) is made up of two words that go together. When the first term is applied, it depicts free oscillations with amplitude A, phase, and frequency = 1, which is known as the natural frequency of the oscillator. The second term is applied to the response to the first data, and it defines a response to the initial data. The second term develops as a result of the external force, and as a result, it depicts the oscillations that are compelled to occur. In order to study some of the most intriguing aspects of solution (4.2.35), we choose the following situations of particular interest:

(i) Zero Forcing Function.

In this case, solution (4.2.35) reduces to

$$x(t) = A\cos(\omega t - \phi). \qquad (4.2.36)$$

This represents simple harmonic motion with amplitude A, frequency ω and phase φ. Evidently, the motion is oscillatory.

(ii) Steady Forcing Function, that is, f(t)=1.

In this case, solution (4.2.35) becomes

$$x - \frac{F}{\omega^2} = A\cos(\omega t - \phi) - \frac{F}{\omega^2}\cos\omega t. \qquad (4.2.37)$$

In particular, when the particle is released from rest, U = 0, (4.2.37) takes the form

$$x - \frac{F}{\omega^2} = \left(a - \frac{F}{\omega^2}\right)\cos\omega t. \qquad (4.2.38)$$

This corresponds to free oscillations with the natural frequency ω and displays a shift in the equilibrium position from the origin to the point $\frac{F}{\omega^2}$.

4.3 PARTIAL DIFFERENTIAL EQUATIONS, INITIAL AND BOUNDARY VALUE PROBLEMS

Several partial differential equations with given beginning and boundary conditions may be solved using the Laplace transform technique, which is particularly effective in this situation. The use of the Laplace transform technique is shown in the following instances.

Example 4.3.1 (First-Order Initial-Boundary Value Problem) Solve the equation

$$u_t + x u_x = x, \qquad x > 0, \quad t > 0, \qquad (4.3.1)$$

With the initial and boundary conditions

$$u(x,0) = 0 \qquad \text{for} \quad x > 0, \qquad (4.3.2)$$
$$u(0,t) = 0 \qquad \text{for} \quad t > 0. \qquad (4.3.3)$$

We apply the Laplace transform of u(x, t) with respect to t to obtain

$$s\,\bar{u}(x,s) + x\frac{d\bar{u}}{dx} = \frac{x}{s}, \qquad \bar{u}(0,s) = 0.$$

Using the integrating factor xs, the solution of this transformed equation is

$$\bar{u}(x,s) = A\,x^{-s} + \frac{x}{s(s+1)},$$

Where A is a constant of integration Since ¯u (0, s)=0, A = 0 for a bounded solution. Consequently,

$$\bar{u}(x,s) = \frac{x}{s(s+1)} = x\left(\frac{1}{s} - \frac{1}{s+1}\right).$$

The inverse Laplace transform gives the solution

$$u(x,t) = x(1 - e^{-t}). \qquad (4.3.4)$$

Example 4.3.2 Find the solution of the equation

$$x\,u_t + u_x = x, \qquad x > 0, \quad t > 0 \qquad (4.3.5)$$

With the same initial and boundary conditions (4.3.2) and (4.3.3).

Application of the Laplace transform with respect to t to (4.3.5) with the initial condition gives

$$\frac{d\bar{u}}{dx} + x\,s\,\bar{u} = \frac{x}{s}.$$

Using the integrating factor exp $\left(\frac{1}{2}x^2 s\right)$ gives the solution

$$\bar{u}(x,s) = \frac{1}{s^2} + A\exp\left(-\frac{1}{2}s\,x^2\right),$$

Where A is an integrating constant Since $\bar{u}(0, s)=0$, A = $\frac{1}{s^2}$ and hence, the solution is

$$\bar{u}(x,s) = \frac{1}{s^2}\left[1 - \exp\left(-\frac{1}{2}x^2 s\right)\right]. \qquad (4.3.6)$$

Finally, we obtain the solution by inversion

$$u(x,t) = t - \left(t - \frac{1}{2}x^2\right) H\left(t - \frac{x^2}{2}\right). \qquad (4.3.7)$$

Or, equivalently,

$$u(x,t) = \left\{ \begin{array}{ll} t, & 2t < x^2 \\ \frac{1}{2}x^2, & 2t > x^2 \end{array} \right\}. \qquad (4.3.8)$$

In this case, 4.3.3 (The Heat Conduction Equation in a Semi-Infinite Medium) Solve the problem in your head.

$$u_t = \kappa\,u_{xx}, \quad x > 0, \quad t > 0 \qquad (4.3.9)$$

With the initial and boundary conditions

$$u(x,0) = 0, \qquad x > 0 \qquad (4.3.10)$$

$$u(0,t) = f(t), \qquad t > 0 \qquad (4.3.11)$$

$$u(x,t) \to 0 \qquad \text{as } x \to \infty, \quad t > 0. \qquad (4.3.12)$$

Application of the Laplace transform with respect to t to (4.3.9) gives

$$\frac{d^2\bar{u}}{dx^2} - \frac{s}{\kappa}\bar{u} = 0. \qquad (4.3.13)$$

The general solution of this equation is

$$\bar{u}(x,s) = A\exp\left(-x\sqrt{\frac{s}{\kappa}}\right) + B\exp\left(x\sqrt{\frac{s}{\kappa}}\right), \qquad (4.3.14)$$

Where A and B are integrating constants. For a bounded solution, B ≡ 0, and using ¯u(0, s) = ¯f(s), we obtain the solution

$$\bar{u}(x,s) = \bar{f}(s)\exp\left(-x\sqrt{\frac{s}{\kappa}}\right). \qquad (4.3.15)$$

The inversion theorem gives the solution

$$u(x,t) = \frac{x}{2\sqrt{\pi\kappa}}\int_0^t f(t-\tau)\tau^{-3/2}\exp\left(-\frac{x^2}{4\kappa\tau}\right)d\tau, \qquad (4.3.16)$$

Which is, by putting? $\lambda = \frac{x}{2\sqrt{\kappa\tau}}$, or, $d\lambda = -\frac{x}{4\sqrt{\kappa}}\tau^{-3/2}d\tau$,

$$= \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{\kappa t}}}^{\infty} f\left(t - \frac{x^2}{4\kappa\lambda^2}\right)e^{-\lambda^2}\,d\lambda.$$

$$(4.3.17)$$

This is the formal solution of the problem. In particular, if f(t) = T0 = constant, solution (4.3.17) becomes

$$u(x,t) = \frac{2T_0}{\sqrt{\pi}} \int_{\frac{x}{\kappa t}}^{\infty} e^{-\lambda^2} d\lambda = T_0 erfc\left(\frac{x}{2\sqrt{\kappa t}}\right). \qquad (4.3.18)$$

As t increases, it is clear that the temperature distribution approaches asymptotically to the constant value T0. On the other hand, we will discuss another physical issue that is concerned with the determination of the temperature distribution in a semi-infinite solid when the rate of heat flow is specified at the end of the solid when x = 0. As a result, the challenge is to solve the diffusion equation (4.3.9) in the presence of the circumstances (4.3.10). (4.3.12)

$$-k\left(\frac{\partial u}{\partial x}\right) = g(t) \quad \text{at } x = 0, \quad t > 0, \qquad (4.3.19)$$

Where k is a constant that is called thermal conductivity Application of the Laplace transform gives the solution of the transformed problem

$$\bar{u}(x,s) = \frac{1}{k}\sqrt{\frac{\kappa}{s}}\, \bar{g}(s) \exp\left(-x\sqrt{\frac{s}{\kappa}}\right). \qquad (4.3.20)$$

The inverse Laplace transform yields the solution

$$u(x,t) = \frac{1}{k}\sqrt{\frac{\kappa}{\pi}} \int_0^t g(t-\tau)\, \tau^{-\frac{1}{2}} \exp\left(-\frac{x^2}{4\kappa t}\right) d\tau, \qquad (4.3.21)$$

Which is, by the change of variable? $\lambda = \frac{x}{2\sqrt{\kappa\tau}}$,

$$= \frac{x}{k\sqrt{\pi}} \int_{\frac{x}{\sqrt{4\kappa t}}}^{\infty} g\left(t - \frac{x^2}{4\kappa\lambda^2}\right) \lambda^{-2} e^{-\lambda^2} d\lambda. \qquad (4.3.22)$$

In particular, if g(t) = T0 = constant, the solution becomes

$$u(x,t) = \left(\frac{T_0 x}{k\sqrt{\pi}}\right) \int_{\frac{x}{\sqrt{4\kappa t}}}^{\infty} \lambda^{-2} e^{-\lambda^2} d\lambda.$$

Integrating this result by parts gives the solution

$$u(x,t) = \frac{T_0}{\kappa}\left[2\sqrt{\frac{kt}{\pi}}\exp\left(-\frac{x^2}{4\kappa t}\right) - x\, erfc\left(\frac{x}{2\sqrt{\kappa t}}\right)\right]. \qquad (4.3.23)$$

Alternatively, the heat conduction problem (4.3.9)–(4.3.12) can be solved by using fractional derivatives (see Chapter 5 or Debnath, 1978). We recall (4.3.15) and rewrite it

$$\frac{\partial \bar{u}}{\partial x} = -\sqrt{\frac{s}{\kappa}}\,\bar{u}. \qquad (4.3.24)$$

In view of (3.9.21), this can be expressed in terms of fractional derivative of order $\frac{1}{2}$ as

$$\frac{\partial u}{\partial x} = -\frac{1}{\sqrt{\kappa}}\mathscr{L}^{-1}\left\{\sqrt{s}\,\bar{u}(x,s)\right\} = -\frac{1}{\sqrt{\kappa}}\,{}_0D_t^{\frac{1}{2}}u(x,t). \qquad (4.3.25)$$

Thus, the heat flux is expressed in terms of the fractional derivative. In particular, when u(0, t) = constant = T0, then the heat flux at the surface is

$$-k\left(\frac{\partial u}{\partial x}\right)_{x=0} = \frac{k}{\sqrt{\kappa}}D_t^{\frac{1}{2}}T_0 = \frac{kT_0}{\sqrt{\pi\kappa\, t}}. \qquad (4.3.26)$$

Example 4.3.4 (Diffusion Equation in a Finite Medium) Solve the diffusion equation

$$u_t = \kappa\, u_{xx}, \quad 0 < x < a, \quad t > 0, \qquad (4.3.27)$$

With the initial and boundary conditions

$$u(x,0) = 0, \quad 0 < x < a, \qquad (4.3.28)$$
$$u(0,t) = U, \quad t > 0, \qquad (4.3.29)$$
$$u_x(a,t) = 0, \quad t > 0, \qquad (4.3.30)$$

Where U is a constant We introduce the Laplace transform of u(x, t) with respect to t to obtain

$$\frac{d^2\bar{u}}{dx^2} - \frac{s}{\kappa}\,\bar{u} = 0, \qquad 0 < x < a, \qquad (4.3.31)$$

$$\bar{u}(0,s) = \frac{U}{s}, \quad \left(\frac{d\bar{u}}{dx}\right)_{x=a} = 0. \qquad (4.3.32\text{ab})$$

The general solution of (4.3.31) is

$$\bar{u}(x,s) = A\cosh\left(x\sqrt{\frac{s}{\kappa}}\right) + B\sinh\left(x\sqrt{\frac{s}{\kappa}}\right), \qquad (4.3.33)$$

Where A and B are constants of integration. Using (4.3.32ab), we obtain the values of A and B so that the solution (4.3.33) becomes

$$\bar{u}(x,s) = \frac{U}{s} \cdot \frac{\cosh\left[(a-x)\sqrt{\frac{s}{\kappa}}\right]}{\cosh\left(a\sqrt{\frac{s}{\kappa}}\right)}. \qquad (4.3.34)$$

The inverse Laplace transform gives the solution

$$u(x,t) = U\mathscr{L}^{-1}\left\{\frac{\cosh(a-x)\sqrt{\frac{s}{\kappa}}}{s\cosh\left(a\sqrt{\frac{s}{\kappa}}\right)}\right\}. \qquad (4.3.35)$$

The inversion can be carried out by the Cauchy Residue Theorem to obtain

$$u(x,t) = U\left[1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos\left\{\frac{(2n-1)(a-x)\pi}{2a}\right\} \times \exp\left\{-(2n-1)^2\left(\frac{\pi}{2a}\right)^2\kappa t\right\}\right], \qquad (4.3.36)$$

which is, by expanding the cosine term,

$$= U\left[1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\left\{\left(\frac{2n-1}{2a}\right)\pi x\right\} \times \exp\left\{-(2n-1)^2\left(\frac{\pi}{2a}\right)^2\kappa t\right\}\right]. \qquad (4.3.37)$$

This result can be obtained by the method of separation of variables.

4.4 SOLUTION OF INTEGRAL EQUATIONS

DEFINITION 4.4.1 an equation in which the unknown function occurs under an integral is called an integral equation.

An equation of the form

$$f(t) = h(t) + \lambda \int_a^b k(t, \tau) f(\tau)\, d\tau, \qquad (4.4.1)$$

This equation, in which f is the unknown function, h(t), k(t,); and the integration limits a and b are known; and is a constant, is referred to as the linear integral equation of the second kind or the linear Volterra integral equation. h(t), k(t,); and a and b are known, is known as the linear Volterra integral equation. The kernel of the equation is defined as the function k(t,) where t is the time in seconds. In the case of h(t)=0 or h(t) = 0, such an equation is said to be homogeneous or inhomogeneous, respectively. Whenever the kernel of an equation takes the form k(t, t) = g(t, t), the equation is referred to as a convolution integral equation, and when it does not, it is known as a convolution integral equation. In this part, we demonstrate how the Laplace transform technique may be used to solve the convolution integral equations with success using the Laplace transform. Exemplifications are available for this strategy, which is uncomplicated and easy to understand.

To solve the convolution integral equation of the form

$$f(t) = h(t) + \lambda \int_0^t g(t-\tau) f(\tau)\, d\tau, \qquad (4.4.2)$$

We take the Laplace transform of this equation to obtain

$$\bar{f}(s) = \bar{h}(s) + \lambda \mathscr{L} \left\{ \int_0^t g(t-\tau) f(\tau)\, d\tau \right\},$$

Which is, by the Convolution

Theorem

$$\bar{f}(s) = \bar{h}(s) + \lambda \bar{f}(s)\bar{g}(s).$$

Or

$$\bar{f}(s) = \frac{\bar{h}(s)}{1 - \lambda \bar{g}(s)}. \qquad (4.4.3)$$

Inversion gives the formal solution

$$f(t) = \mathscr{L}^{-1}\left\{\frac{\bar{h}(s)}{1 - \lambda \bar{g}(s)}\right\}. \qquad (4.4.4)$$

In many simple cases, the right-hand side can be inverted by using partial fractions or the theory of residues. Hence, the solution can readily be found.

Example 4.4.1 Solve the integral equation

$$f(t) = a + \lambda \int_0^t f(\tau)\, d\tau. \qquad (4.4.5)$$

We take the Laplace transform of (4.4.5) to find

$$\bar{f}(s) = \frac{a}{s - \lambda}.$$

Whence, by inversion, it follows that

$$f(t) = a \exp(\lambda t). \qquad (4.4.6)$$

Example 4.4.2 Solve the integro-differential equation

$$f(t) = a \sin t + 2 \int_0^t f'(\tau) \sin(t - \tau)\, d\tau, \quad f(0) = 0. \qquad (4.4.7)$$

Taking the Laplace transform, we obtain

$$\bar{f}(s) = \frac{a}{s^2+1} + 2\mathscr{L}\{f'(t)\}\mathscr{L}\{\sin t\}$$

Or

$$\bar{f}(s) = \frac{a}{s^2+1} + 2\frac{\{s\bar{f}(s) - f(0)\}}{s^2+1}.$$

Hence, by the initial condition,

$$\bar{f}(s) = \frac{a}{(s-1)^2}.$$

Inversion yields the solution

$$f(t) = a\,t\exp(t). \qquad (4.4.8)$$

Example 4.4.3 Solve the integral equation

$$f(t) = a\,t^n - e^{-bt} - c\int_0^t f(\tau)\,e^{c(t-\tau)}\,d\tau. \qquad (4.4.9)$$

Taking the Laplace transform, we obtain

$$\bar{f}(s) = \frac{a\,n!}{s^{n+1}} - \frac{1}{s+b} - \bar{f}(s)\frac{c}{s-c}$$

So that we have

$$\begin{aligned}
\bar{f}(s) &= \left(\frac{s-c}{s}\right)\left[\frac{a\,n!}{s^{n+1}} - \frac{1}{s+b}\right] \\
&= \frac{a\,n!}{s^{n+1}} - \frac{(ac)n!}{s^{n+2}} - \frac{1}{s}\left[\frac{s+b-c-b}{s+b}\right] \\
&= \frac{a\,n!}{s^{n+1}} - \frac{(ac)n!}{s^{n+2}} - \frac{1}{s} + \frac{c+b}{b}\left[\frac{1}{s} - \frac{1}{s+b}\right] \\
&= \frac{a\,n!}{s^{n+1}} - \frac{(ac)n!}{s^{n+2}} - \frac{1}{s} + \left(1+\frac{c}{b}\right)\frac{1}{s} - \left(1+\frac{c}{b}\right)\frac{1}{s+b} \\
&= \frac{a\,n!}{s^{n+1}} - \frac{(ac)n!}{s^{n+2}} + \frac{c}{bs} - \left(1+\frac{c}{b}\right)\frac{1}{s+b}.
\end{aligned}$$

Inversion yields the solution

$$f(t) = at^n - \frac{n!ac}{(n+1)!}t^{n+1} + \frac{c}{b} - \left(1 + \frac{c}{b}\right)e^{-bt}.$$

Example 4.4.4 Find the solution of an integral equation for subsidence model in mining operations (see J. H. Giese, SIAM Rev. 5, (1963) 1–6):

$$f(t) = \frac{1}{(1+t)^2} + a\int_0^t \frac{f(\tau)}{(1+t-\tau)^2}d\tau. \qquad (4.4.10)$$

The integral equation has the convolution form

$$f(t) = \frac{1}{(1+t)^2} + a\, f(t) * \frac{1}{(1+t)^2}. \qquad (4.4.11)$$

Application of the Laplace transforms yields

$$\bar{f}(s) = \mathscr{L}\left\{\frac{1}{(1+t)^2}\right\} + a\,\bar{f}(s)\mathscr{L}\left\{\frac{1}{(1+t)^2}\right\}, \qquad (4.4.12)$$

Where

$$\begin{aligned}\mathscr{L}\left\{\frac{1}{(1+t)^2}\right\} &= -\mathscr{L}\left\{\frac{d}{dt}\frac{1}{1+t}\right\} = 1 - s\,\mathscr{L}\left\{\frac{1}{1+t}\right\} \\ &= 1 - s\int_0^\infty e^{-st}\frac{dt}{1+t}, \qquad \text{put } u = s(1+t) \\ &= 1 - s\,e^s\int_s^\infty \frac{e^{-u}}{u}du = 1 - s\,e^s\,Ei(s) = \bar{k}(s), \quad (4.4.13)\end{aligned}$$

Where Ei(s) is the exponential integral defined in problem 22(b) in 3.9 Exercises. Consequently, (4.4.12) gives

$$\bar{f}(s) = \bar{k}(s) + a\,\bar{f}(s)\bar{k}(s)$$

So that

$$\bar{f}(s) = \frac{\bar{k}(s)}{1 - a\,\bar{k}(s)}. \qquad (4.4.14)$$

The inverse Laplace transform gives the solution

$$f(t) = \mathscr{L}^{-1}\left\{\frac{\bar{k}(s)}{1 - a\,\bar{k}(s)}\right\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\,\frac{\bar{k}(s)}{1 - a\,\bar{k}(s)}\,ds, \qquad (4.4.15)$$

Where c > 0 is taken so that all zeros of the [1 − a ¯ k(s)] are located on the left side of the contour In order to get more in-depth information regarding evaluating the complex integral, the reader is recommended to Giese (1963) as well as L. Fox and E. J. Goodwin, Phil. Roy. Soc. Lond. 245 (1953), pages 501–530.

Example 4.4.5 Solve the integral equation

$$f(t) + a\int_0^t f(\tau)\,e^{a(t-\tau)}d\tau = g(t). \qquad (4.4.16)$$

We apply the Laplace transform to (4.4.16) to obtain the solution for ¯f(s) in the form

$$\bar{f}(s) = \left(1 - \frac{a}{s}\right)\bar{g}(s). \qquad (4.4.17)$$

Application of the inverse Laplace transform gives

$$f(t) = g(t) - a\int_0^t g(\tau)\,d\tau. \qquad (4.4.18)$$

4.5 SOLUTION OF BOUNDARY VALUE PROBLEMS

Aside from finding solutions to basic boundary value issues that occur in many fields of applied mathematics and engineering sciences, the Laplace transform approach is very valuable in many other areas as well. We use the approach to solve boundary value issues in the theory of deflection of elastic beams to demonstrate its effectiveness. When the combined effects of the beam's own weight and the imposed load on the beam are considered, a horizontal beam suffers a vertical deflection. We examine a beam of length, and the equilibrium position of the beam is obtained along the horizontal x-axis of the beam.

Example 4.5.1 (Deflection of Beams)

The differential equation for the vertical deflection y(x) of a uniform beam under the action of a transverse load W(x) per unit length at a distance x from the origin on the x-axis of the beam is

$$EI\frac{d^4y}{dx^4} = W(x), \qquad \text{for} \quad 0 < x < \ell, \qquad (4.5.1)$$

, where E is the Young's modulus, I is the moment of inertia of the cross section around an axis normal to the plane of bending, and EI is referred to as the flexural stiffness of the beam Some physical variables connected with the issue are y (x), M(x) = EIy (x), and S(x) = M (x) = EIy (x), which indicate, respectively, the slope, bending moment, and shear at a location in the coordinate system. Discovering the solution of (4.5.1) under a given loading function and simple boundary conditions including deflection, slope, bending moment and shear is of particular interest. We take into consideration the following scenarios:

(i) Concentrated load on a clamped beam of length , that is, W(x) ≡ W δ(x − a), y(0) = y (0) = 0 and y() = y ()=0, where W is a constant and 0 <a<.

(ii) Distributed load on a uniform beam of length clamped at x = 0 and unsupported at x = , that is, W(x) = W H(x − a), y(0) = y0) = 0, and M() = S() = 0.

(iii) A uniform semi-infinite beam freely hinged at x = 0 resting horizontally on an elastic foundation and carrying a load W per unit length.

In order to solve the problem, we use the Laplace transform ¯y(s) of y(x) defined by

$$\bar{y}(s) = \int_0^\infty e^{-sx} y(x)\, dx. \qquad (4.5.2)$$

In view of this transformation, equation (4.5.1) becomes

$$EI[s^4\bar{y}(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] = \overline{W}(s). \qquad (4.5.3)$$

The solution of the transformed deflection function ¯y(s) for case (i) is

$$\bar{y}(s) = \frac{y''(0)}{s^3} + \frac{y'''(0)}{s^4} + \frac{W}{EI}\frac{e^{-as}}{s^4}. \qquad (4.5.4)$$

Inversion gives

$$y(x) = y''(0)\frac{x^2}{2} + y'''(0)\frac{x^3}{6} + \frac{W}{6EI}(x-a)^3 H(x-a). \qquad (4.5.5)$$

$$y'(x) = y''(0)x + \frac{1}{2}x^2 y'''(0) + \frac{W}{2EI}(x-a)^2 H(x-a). \qquad (4.5.6)$$

The conditions y() = y () = 0 require that

$$y''(0)\frac{\ell^2}{2} + y'''(0)\frac{\ell^3}{6} + \frac{W}{6EI}(\ell-a)^3 = 0,$$

$$y''(0)\ell + y'''(0)\frac{\ell^2}{2} + \frac{W}{2EI}(\ell-a)^2 = 0.$$

These algebraic equations determine the value of y (0) and y (0). Solving these equations, it turns out that

$$y''(0) = \frac{Wa(\ell-a)^2}{EI\,\ell^2} \quad \text{and} \quad y'''(0) = -\frac{W(\ell-a)^2(\ell+2a)}{EI\,\ell^3}.$$

Thus, the final solution for case (i) is

$$y(x) = \frac{W}{2EI}\left[\frac{a(\ell-a)^2 x^2}{\ell^2} - \frac{(\ell-a)^2(\ell+2a)x^3}{3\ell^3} + \frac{(x-a)^3 H(x-a)}{3}\right]. \qquad (4.5.7)$$

It is now possible to calculate the bending moment and shear at any point of the beam, and, in particular, at the ends.

The solution for case (ii) follows directly from (4.5.3) in the form

$$\overline{y}(s) = \frac{y''(0)}{s^3} + \frac{y'''(0)}{s^4} + \frac{W}{EI}\frac{e^{-as}}{s^5}. \qquad (4.5.8)$$

The inverse transformation yields

$$y(x) = \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{W}{24EI}(x-a)^4 H(x-a), \qquad (4.5.9)$$

Where y (0) and y (0) are to be determined from the remaining boundary conditions M() = S() = 0, that is, y () = y (t) = 0.

From (4.5.9) with y () = y () = 0, it follows that

$$y''(0) + y'''(0)\ell + \frac{W}{2EI}(\ell - a)^2 = 0$$
$$y'''(0) + \frac{W}{EI}(\ell - a) = 0$$

Which give

$$y''(0) = \frac{W(\ell - a)(\ell + a)}{2EI} \text{ and } y'''(0) = -\frac{W}{EI}(\ell - a).$$

Hence, the solution for y(x) for case (ii) is

$$y(x) = \frac{W}{2EI}\left[\frac{(\ell^2 - a^2)x^2}{2} - (\ell - a)\frac{x^3}{3} + \frac{W}{12}(x - a)^4 H(x - a)\right]. \qquad (4.5.10)$$

The shear, S, and the bending moment, M, at the origin, can readily be calculated from the solution. The differential equation for case (iii) takes the form

$$EI\frac{d^4y}{dx^4} + ky = W, \quad x > 0, \qquad (4.5.11)$$

k is a positive constant, and the influence of elastic foundation is represented by the second component on the left-hand side of the equation.

Writing $\left(\frac{k}{EI}\right) = 4\omega^4$, equation (4.5.11) becomes

$$\left(\frac{d^4}{dx^4} + 4\omega^4\right) y(x) = \frac{W}{EI}, \quad x > 0. \qquad (4.5.12)$$

This has to be solved subject to the boundary conditions

$$y(0) = y''(0) = 0, \qquad (4.5.13)$$
$$y(x) \text{ is finite as } x \to \infty. \qquad (4.5.14)$$

Using the Laplace transform with respect to x to (4.5.12), we obtain

$$(s^4 + 4\omega^4)\,\bar{y}(s) = \left(\frac{W}{EI}\right)\frac{1}{s} + sy'(0) + y'''(0). \qquad (4.5.15)$$

In view of the Tauberian Theorem 3.8.2 (ii), that is,

$$\lim_{s\to 0} s\,\bar{y}(s) = \lim_{x\to\infty} y(x).$$

it follows that ¯y(s) must be of the form

$$\bar{y}(s) = \frac{W}{EI}\frac{1}{s(s^4+4\omega^4)}, \qquad (4.5.16)$$

Which gives?

$$\lim_{x\to\infty} y(x) = \frac{W}{k}. \qquad (4.5.17)$$

We now write (4.5.16) as

$$\bar{y}(s) = \frac{W}{EI}\frac{1}{4\omega^4}\left[\frac{1}{s} - \frac{s^3}{s^4+4\omega^4}\right]. \qquad (4.5.18)$$

Using the standard table of inverse Laplace transforms, we obtain

$$\begin{aligned} y(x) &= \frac{W}{k}(1-\cos\omega x\,\cosh\omega x) \\ &= \frac{W}{k}\left[1-\frac{1}{2}e^{-\omega x}\cos\omega x - \frac{1}{2}e^{\omega x}\cos\omega x\right]. \qquad (4.5.19) \end{aligned}$$

In view of (4.5.17), the final solution is

$$y(x) = \frac{W}{k}\left(1-\frac{1}{2}e^{-\omega x}\cos\omega x\right). \qquad (4.5.20)$$

4.6 EVALUATION OF DEFINITE INTEGRALS

The Laplace transform may be used to assess certain definite integrals including a parameter with ease when the parameter is known. Despite the fact that the assessment approach is not very rigorous, it is basic and uncomplicated. It is primarily predicated on the possibility of switching the order of integration, which is to say that the approach is interchangeable.

$$\mathscr{L}\int_a^b f(t,x)\,dx = \int_a^b \mathscr{L}f(t,x)\,dx, \qquad (4.6.1)$$

And may be well described by considering some important integrals.

Example 4.6.1 Evaluate the integral

$$f(t) = \int_0^\infty \frac{\sin tx}{x(a^2+x^2)}\,dx. \qquad (4.6.2)$$

With regard to t, we take the Laplace transform of (4.6.2) and exchange the order of integration, which is possible owing to uniform convergence, to get

$$\begin{aligned}
\bar{f}(s) &= \int_0^\infty \frac{dx}{x(a^2+x^2)} \int_0^\infty e^{-st}\sin tx\,dt \\
&= \int_0^\infty \frac{dx}{(a^2+x^2)(x^2+s^2)} \\
&= \frac{1}{s^2-a^2}\int_0^\infty \left(\frac{1}{a^2+x^2} - \frac{1}{x^2+s^2}\right)dx \\
&= \frac{1}{s^2-a^2}\left(\frac{1}{a} - \frac{1}{s}\right)\frac{\pi}{2} \\
&= \frac{\pi}{2}\frac{1}{s(s+a)} = \frac{\pi}{2}\left(\frac{1}{s} - \frac{1}{s+a}\right).
\end{aligned}$$

Inversion gives the value of the given integral

$$f(t) = \frac{\pi}{2a}(1-e^{-at}). \qquad (4.6.3)$$

Example 4.6.2 Evaluate the integral

$$f(t) = \int_0^\infty \frac{\sin^2 tx}{x^2}\,dx. \qquad (4.6.4)$$

A procedure similar to the above integral with 2 sin2 tx = 1 − cos(2 tx) gives

$$\bar{f}(s) = \frac{1}{2}\int_0^\infty \frac{1}{x^2}\left(\frac{1}{s} - \frac{s}{4x^2+s^2}\right)dx = \frac{2}{s}\int_0^\infty \frac{dx}{4x^2+s^2}$$
$$= \frac{1}{s}\int_0^\infty \frac{dy}{y^2+s^2} = \frac{1}{s^2}\left[\tan^{-1}\frac{y}{s}\right]_0^\infty = \pm\frac{\pi}{2s^2}$$

According as s > or < 0. The inverse transforms yields

$$f(t) = \frac{\pi t}{2}\, sgn\, t. \qquad (4.6.5)$$

Example 4.6.3 Show that

$$\int_0^\infty \frac{x \sin xt}{x^2+a^2}\,dx = \frac{\pi}{2}e^{-at}, \quad (a, t > 0). \qquad (4.6.6)$$

Suppose

$$f(t) = \int_0^\infty \frac{x \sin xt}{x^2+a^2}\,dx.$$

Taking the Laplace transform with respect to t gives

$$\bar{f}(s) = \int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+s^2)}$$
$$= \int_0^\infty \frac{dx}{x^2+s^2} - \frac{a^2}{s^2-a^2}\int_0^\infty \left(\frac{1}{x^2+a^2} - \frac{1}{x^2+s^2}\right)dx$$
$$= \frac{\pi}{2s}\left(1 - \frac{a}{s+a}\right) = \frac{\pi}{2}\frac{1}{(s+a)}.$$

Taking the inverse transform, we obtain

$$f(t) = \frac{\pi}{2}e^{-at}.$$

4.7 APPLICATION OF THE JOINED LAPLACE AND FOURIER TRANSFORMS

Example 4.8.1 (The Inhomogeneous Cauchy Problem for the Wave Equation)

In order to solve the Cauchy issue for the wave equation as mentioned in Example 2.12.4, the combined Fourier and Laplace transform approach should be used in conjunction with an inhomogeneous term, q. (x, t). The combined Fourier and Laplace transform of u(x, t) is denoted by the symbol

$$\bar{U}(k,s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\,dx \int_{0}^{\infty} e^{-st} u(x,t)\,dt. \qquad (4.8.1)$$

The transformed inhomogeneous Cauchy problem has the solution in the form

$$\bar{U}(k,s) = \frac{sF(k) + G(k) + \bar{Q}(k,s)}{(s^2 + c^2k^2)}, \qquad (4.8.2)$$

Where $\bar{Q}$(k, s) is the joint transform of the inhomogeneous term, q(x, t) present on the right side of the wave equation.

The joint inverse transform gives the solution as

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \mathscr{L}^{-1}\left[\frac{sF(k) + G(k) + \bar{Q}(k,s)}{s^2 + c^2k^2}\right] dk \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[F(k)\cos ckt + \frac{G(k)}{ck}\sin ckt\right] e^{ikx}\,dk \\ &\qquad + \frac{1}{ck}\int_0^t \sin ck(t-\tau)\, Q(k,\tau)\,d\tau \\ &= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)(e^{ickt} + e^{-ickt}) e^{ikx}\,dk \\ &\qquad + \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{G(k)}{ick}(e^{ickt} - e^{-ickt}) e^{ikx}\,dk \end{aligned}$$

$$+\frac{1}{\sqrt{2\pi}}\frac{1}{2c}\int_0^t d\tau \int_{-\infty}^{\infty}\frac{Q(k,\tau)}{ik}\left[e^{ick(t-\tau)}+e^{-ick(t-\tau)}\right]e^{ikx}dk$$

$$=\frac{1}{2}[f(x-ct)+f(x+ct)]+\frac{1}{\sqrt{2\pi}}\frac{1}{2c}\int_{-\infty}^{\infty}G(k)\,dk\int_{x-ct}^{x+ct}e^{ik\xi}\,d\xi$$

$$+\frac{1}{2c}\int_0^t d\tau\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}Q(k,\tau)\,dk\int_{x-c(t-\tau)}^{x+c(t-\tau)}e^{ik\xi}\,d\xi$$

$$=\frac{1}{2}[f(x-ct)+f(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(\xi)\,d\xi$$

$$+\frac{1}{2c}\int_0^t d\tau\int_{x-c(t-\tau)}^{x+c(t-\tau)}q(\xi,\tau)\,d\xi. \qquad (4.8.3)$$

This is identical with the d'Alembert solution (2.12.41) when q(x, t) ≡ 0.

As an example, Section 4.8.2 (Dispersive Long Water Waves in a Rotating Ocean). A rotating inviscid ocean is investigated using a combined Laplace and Fourier transform to solve linearized horizontal equations of motion and the continuity equation, which are both linearized. Using a rotating coordinate system (see Proudman, 1953; Debnath and Kulchar, 1972), the following equations may be found:

$$\frac{\partial \mathbf{u}}{\partial t}+f\,\hat{\mathbf{k}}\times\mathbf{u}=-\frac{1}{\rho}\nabla p+\frac{1}{\rho h}\boldsymbol{\tau}, \qquad (4.8.4)$$

$$\nabla\cdot\mathbf{u}=-\frac{1}{h}\frac{\partial \zeta}{\partial t}, \qquad (4.8.5)$$

In this equation, u = (u, t) represents the horizontal velocity field, k is the unit vector normal to the horizontal plane, f = 2 sin t represents the constant Coriolis parameter, t represents the vertical free surface elevation, and x and y represent the components of wind stress in the x and y directions, respectively. The pressure is calculated using the hydrostatic equation.

$$p=p_0+g\rho(\zeta-z), \qquad (4.8.6)$$

Where z is the depth of water below the mean free surface and g is the acceleration due to gravity. Equation (4.8.4)–(4.8.5) combined with (4.8.6) reduce to the form

$$\frac{\partial u}{\partial t} - f\upsilon = -g\frac{\partial \zeta}{\partial x} + \frac{\tau^x}{\rho h}, \quad (4.8.7)$$

$$\frac{\partial \upsilon}{\partial t} + fu = -g\frac{\partial \zeta}{\partial y} + \frac{\tau^y}{\rho h}, \quad (4.8.8)$$

$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} = -\frac{1}{h}\frac{\partial \zeta}{\partial t}. \quad (4.8.9)$$

It follows from (4.8.7)–(4.8.8) that

$$Du = -g\left(\frac{\partial^2}{\partial x \partial t} + f\frac{\partial}{\partial y}\right)\zeta + \frac{1}{\rho h}\left(\frac{\partial \tau^x}{\partial t} + f\tau^y\right), \quad (4.8.10)$$

$$D\upsilon = -g\left(\frac{\partial^2}{\partial y \partial t} - f\frac{\partial}{\partial x}\right)\zeta + \frac{1}{\rho h}\left(\frac{\partial \tau^y}{\partial t} - f\tau^x\right), \quad (4.8.11)$$

Where the differential operator D is

$$D \equiv \left(\frac{\partial^2}{\partial t^2} + f^2\right). \quad (4.8.12)$$

Elimination of u and υ from (4.8.9)–(4.8.11) gives

$$\left(\nabla^2 - \frac{1}{c^2}D\right)\zeta_t = E(x, y, t), \quad (4.8.13)$$

Where c2 = gh and ∇2 is the horizontal Laplacian, and E(x, y, t) is a known forcing function given by

$$E(x, y, t) = \frac{1}{\rho c^2}\left[\frac{\partial^2 \tau^x}{\partial x \partial t} + \frac{\partial^2 \tau^y}{\partial y \partial t} + f\left(\frac{\partial \tau^y}{\partial x} - \frac{\partial \tau^x}{\partial y}\right)\right]. \quad (4.8.14)$$

Furthermore, we assume that the circumstances are homogeneous in the y direction and that the wind stress operates only in the x direction, resulting in x and E being provided as functions of x and t alone, respectively. As a result, the equation (4.8.13) is obtained.

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2} + f^2\right)\right]\zeta_t = \frac{1}{\rho c^2}\left(\frac{\partial^2 \tau^x}{\partial x \partial t}\right).$$

Integrating this equation with respect to t gives

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2} + f^2\right)\right]\zeta = \frac{1}{\rho c^2}\left(\frac{\partial \tau^x}{\partial x}\right). \qquad (4.8.15)$$

Similarly, the velocity u(x, t) satisfies the equation

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2} + f^2\right)\right]u = -\frac{1}{\rho h c^2}\left(\frac{\partial \tau^x}{\partial t}\right). \qquad (4.8.16)$$

Because they include zero on the right-hand side, equations (4.8.15) and (4.8.16) are known as the Klein-Gordon equations, which have garnered a great deal of attention in both quantum physics and applied mathematics. The following boundary and beginning conditions must be met in order to solve the equation (4.8.15):

$$|\zeta| \quad \text{is bounded as } |x| \to \infty, \qquad (4.8.17)$$

$$\zeta(x,t) = 0 \quad \text{at} \quad t = 0 \text{ for all real } x. \qquad (4.8.18)$$

The solution of the homogeneous equation (4.8.15) in the form of a plane wave is sought before attempting to solve the starting value issue.

$$\zeta(x,t) = A\exp\{i(\omega t - kx)\}, \qquad (4.8.19)$$

Where A is a constant amplitude, ω is the frequency, and k is the wavenumber. Such a solution exists provided the dispersion relation

$$\omega^2 = c^2k^2 + f^2 \qquad (4.8.20)$$

Is satisfied. Thus, the phase and the group velocities of waves are given by

$$C_p = \frac{\omega}{k} = \left(c^2 + \frac{f^2}{k^2}\right)^{\frac{1}{2}}, \quad C_g = \frac{\partial \omega}{\partial k} = \frac{c^2 k}{(c^2k^2 + f^2)^{\frac{1}{2}}}. \qquad (4.8.21ab)$$

As a result, the waves in a spinning ocean (f = 0) are dispersive in nature. In contrast, in a nonrotating ocean (f = 0), all waves would travel with constant velocity c, and they would be non-dispersive shallow water waves, as opposed to rotational waves. Furthermore, since CpCg = c2, it follows that the phase velocity has a minimum value of c and the group velocity has a maximum value of c. Even though they have the shortest phase velocity, the short waves will be seen first at a given location. The transformed solution is obtained by applying the combined Laplace and Fourier transform on (4.8.15) in conjunction with (4.8.17)–(4.8.18) and getting the transformed solution.

$$\bar{\bar{\zeta}}(k,s) = -\frac{Ac^2}{(s^2+a^2)}\bar{\bar{f}}(k,s), \qquad a^2 = (c^2k^2+f^2), \qquad (4.8.22)$$

Where

$$f(x,t) = \frac{1}{\rho c^2}\left(\frac{\partial \tau^x}{\partial x}\right)H(t). \qquad (4.8.23)$$

The inverse transforms combined with the Convolution Theorem of the Laplace transform lead to the formal solution

$$\zeta(x,t) = -\frac{Ac}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(k^2+\frac{f^2}{c^2}\right)^{-\frac{1}{2}} e^{ikx}\,dk\int_0^t \bar{f}(k,t-\tau)\sin a\tau\,d\tau. \qquad (4.8.24)$$

Generalized speaking, this integral cannot be calculated without first prescribing f(x, t). Even if a specific form of f is provided, obtaining a precise assessment of (4.8.24) is an almost insurmountable challenge. Because of this, it is required to use asymptotic approaches to solve the problem (see Debnath and Kulchar, 1972). The answer is investigated via the use of a specific type of the wind stress distribution.

$$\frac{\tau^x}{\rho c^2} = A\,e^{i\omega t}H(t)H(-x), \qquad (4.8.25)$$

Where A is a constant and ω is the frequency of the applied disturbance. Thus,

$$\frac{1}{\rho c^2}\left(\frac{\partial \tau^x}{\partial x}\right) = -Ae^{i\omega t}H(t)\delta(-x). \qquad (4.8.26)$$

In this case, solution (4.8.24) reduces to the form

$$\zeta(x,t) = \frac{Ac}{\sqrt{2\pi}} \int_0^t e^{i\omega(t-\tau)} H(t-\tau) \mathscr{F}^{-1}\left[\frac{\sin a\tau}{\sqrt{k^2+\frac{f^2}{c^2}}}\right] d\tau$$

$$= \frac{Ac}{2} \int_0^t e^{i\omega(t-\tau)} H(t-\tau) J_0\left\{\frac{f}{c}(c^2\tau^2 - x^2)^{\frac{1}{2}}\right\} \times H(c\tau - |x|)\, d\tau, \qquad (4.8.27)$$

Where J0(z) is the zero-order Bessel function of the first type and z is the value of the function. When = 0, this answer is identical to that found by Crease (1956), who used the Green's function approach to arrive at the solution. Therefore, the answer is as follows:

$$\zeta = \frac{Ac}{2} \int_0^t H(t-\tau) J_0\left[f\left\{\tau^2 - \frac{x^2}{c^2}\right\}^{\frac{1}{2}}\right] H\left(\tau - \frac{|x|}{c}\right) d\tau. \qquad (4.8.28)$$

In terms of non-dimensional parameters f τ = α, f t = a, and fx c = b, solution (4.8.28) assumes the form

$$\left(\frac{2f}{Ac}\right)\zeta = \int_0^a H(a-\alpha) J_0\left[(\alpha^2 - b^2)^{\frac{1}{2}}\right] H(\alpha - |b|)\, d\alpha. \qquad (4.8.29)$$

Or, equivalently,

$$\left(\frac{2f}{Ac}\right)\zeta = \int_{|b|}^d J_0\left[(\alpha^2 - b^2)^{\frac{1}{2}}\right] d\alpha, \qquad (4.8.30)$$

Where d = max (|b|, a). This is the basic solution of the problem. In order to find the solution of (4.8.16), we first choose.

$$\frac{1}{\rho c^2}\left(\frac{\partial \tau^x}{\partial t}\right) = A\delta(t) H(-x), \qquad (4.8.31)$$

So that the joint Laplace and Fourier transform of this result is AF{H(−x)}. Thus, the transformed solution of (4.8.16) is

$$\bar{\bar{u}}(k,s) = \frac{Ac^2}{h}\mathscr{F}\{H(-x)\}\frac{1}{(s^2+\omega^2)}, \quad \omega^2 = (ck)^2 + f^2. \tag{4.8.32}$$

The inverse transforms combined with the Convolution Theorem lead to the solution

$$u(x,t) = \frac{Ac}{2h}\int_{-\infty}^{\infty} H(-\xi)\, J_0\left[f\left\{t^2 - \left(\frac{x-\xi}{c}\right)^2\right\}^{\frac{1}{2}}\right] \times H\left(t - \frac{(x-\xi)}{c}\right) d\xi, \tag{4.8.33}$$

$$u(x,t) = \frac{Ac}{2h}\int_{-\infty}^{\infty} H(-\xi)\, J_0\left[f\left\{t^2 - \left(\frac{x-\xi}{c}\right)^2\right\}^{\frac{1}{2}}\right] \times H\left(t - \frac{(x-\xi)}{c}\right) d\xi, \tag{4.8.33}$$

Which is, by the change of variable $(x - \xi)f = c\alpha$, with $a = f\,t$ and $b = (fx/c)$,

$$= \frac{Ac^2}{2hf}\int_{b}^{\infty} J_0\left[(a^2-\alpha^2)^{\frac{1}{2}}\right] H(a - |\alpha|)\, d\alpha. \tag{4.8.34}$$

For the case $b > 0$, solution (4.8.34) becomes

$$u(x,t) = \frac{Ac^2}{2hf}H(a-b)\int_{b}^{a} J_0\left\{(a^2-\alpha^2)^{\frac{1}{2}}\right\} d\alpha. \tag{4.8.35}$$

When $b < 0$, the velocity field is

$$\begin{aligned} u(x,t) &= \frac{Ac^2}{2hf}\left[\int_{-a}^{a} J_0\left\{(a^2-\alpha^2)^{\frac{1}{2}}\right\} d\alpha - H(a-|b|)\int_{-a}^{b} J_0\left\{(a^2-\alpha^2)^{\frac{1}{2}}\right\} d\alpha\right] \\ &= \frac{gA}{2f}\left[2\sin a - H(a-|b|)\int_{|b|}^{a} J_0\left\{(a^2-\alpha^2)^{\frac{1}{2}}\right\} d\alpha\right], \end{aligned} \tag{4.8.36}$$

Which is, for $a < |b|$,

$$u(x,t) = \left(\frac{gA}{2f}\right)\sin a. \tag{4.8.37}$$

Finally, it can be shown that the velocity transverse to the direction of propagation is

$$v = \left(-\frac{gA}{2f}\right) \int_0^a d\beta \int_b^\infty J_0\left\{(\beta^2 - \alpha^2)^{\frac{1}{2}}\right\} H(\beta - |\alpha|)\, d\alpha. \qquad (4.8.38)$$

If b > 0, that is, x is outside the generating region, then

$$\left(\frac{2f}{gA}\right) v = -H(a-b) \int_b^a d\beta \int_b^\beta J_0\left\{(\beta^2 - \alpha^2)^{\frac{1}{2}}\right\} d\alpha,$$

Which becomes, after some simplification?

$$= -\left[(1 - \cos a) - \int_0^b d\alpha \int_\alpha^a J_0\left\{(\beta^2 - \alpha^2)^{\frac{1}{2}}\right\}\right] H(a-b). \qquad (4.8.39)$$

For b < 0, it is necessary to consider two cases: (i) a < |b| and (ii) a > |b|. In the former case, (4.8.38) takes the form

$$\left(\frac{2f}{gA}\right) v = -\int_0^a d\beta \int_{-\beta}^\beta J_0\left\{(\beta^2 - \alpha^2)^{\frac{1}{2}}\right\} d\alpha = -2(1 - \cos b). \qquad (4.8.40)$$

In the latter case, the final form of the solution is

$$\left(\frac{2f}{gA}\right) v = -(1 - \cos b) + \int_0^{|b|} d\alpha \int_\alpha^a J_0\left\{(\beta^2 - \alpha^2)^{\frac{1}{2}}\right\} d\beta. \qquad (4.8.41)$$

Finally, the steady-state solutions are obtained in the limit as $t \to \infty (b \to \infty)$

$$\begin{aligned}
\zeta &= \frac{Ac}{2f}\exp(-|b|),\\
u &= \frac{Ag}{2f}\sin ft,\\
v &= \frac{Ag}{2f}\begin{bmatrix}\cos ft - \exp(-b), & b>0\\ \cos ft + \exp(-|b|) - 2, & b<0\end{bmatrix}.
\end{aligned} \qquad (4.8.42)$$

Thus, the steady-state solutions are attained in a rotating ocean. This shows a striking contrast with the corresponding solutions in the non-rotating ocean where an ever-increasing free surface elevation is found. The terms sin f t and cos f t involved in the steady-state velocity field represent inertial oscillations with frequency f.

CHAPTER 5

SUMUDU TRANSFORMS AND THEIR PROPERTIES

5.1 INTRODUCTION

The Sumudu transform has already shown great promise, owing to its straightforward construction and the unusual and valuable qualities that result as a result. It has been shown here and elsewhere that it may be used to assist in the solution of complex issues in engineering mathematics and applied science. Despite the promise offered by this new operator, only a few theoretical studies have been published in the literature during a fifteen-year period, despite the possibility offered by this new operator. The Sumudu transform is not mentioned in most, if not all, of the transform theory texts that are currently accessible. No mention of the Sumudu transform can be found in any of the more recent well-known comprehensive handbooks, including. Perhaps this is due to the fact that no transform with this name (in the traditional sense) was announced until the late 1980s and early 1990s of the previous century. On the other hand, it is important to note that an analogous formulation, known as the s-multiplied Laplace transform, was disclosed as early as 1948 (see, for example, and references), if not before. The chapter, which demonstrated the application of the Sumudu transform to partial differential equations, came soon after important work. It was shown in work that the Sumudu transform may be utilised to solve ordinary differential equations and engineering control problems with high accuracy. Another step forward was taken who introduced a complicated inversion formula for the Sumudu transforms, building on Watugala's previous work (see Theorem 3.1 in Section 3).

The transform was presented in a very recent series on how to solve integrodifferential equations, with a particular emphasis on dynamic systems, as well as how to apply the transform to solve differential equations. Using partial differential equations and solutions to partial differential equations as a starting point, Watugala extended the transform to include two variables. Several applications of convolution type integral equations have been illustrated, with a special focus on issues encountered in the manufacturing business. In the preceding part, we discussed how the Laplace-Sumudu duality was used to establish or confirm the usefulness of this unique transform's key characteristics and properties. A two-page table including the transforms of some of the main functions would serve as an example of this kind of presentation. In order to give a more comprehensive list of functions, we include Sumudu transformations at the end of this chapter, which are analogous to the vast Laplace transform list found in Spiegel [9]. A paradigm shift in the conceptual process of Sumudu transform application to differential equations has been achieved through the introduction of broader shift theorems, which appear to have combinatorial connections to generalised stirling numbers. This is in addition to a paradigm shift in the conceptual process of Sumudu transform application to differential equations. A notable breakthrough over earlier work is that we show more general Sumudu differentiation, integration, and convolution theorems than have previously been proven in the literature. The Laplace-Sumudu duality (LSD) is also used in the generation of the complex inverse Sumudu transform, which may be described mathematically as a formula for the Bromwhich contour integral.

5.2 DEFINITION OF THE SUMUDU INTEGRAL AND EXAMPLES

Over the set of functions,

$$A = \{f(t) \mid \exists M, \tau_1, \tau_2 > 0, \left| f(t) \right| < Me^{|t|/\tau_j}, \text{ if } t \in (-1)^j \times [0, \infty)\},$$

The Sumudu transform is defined by

$$G(u) = \mathbb{S}[f(t)] = \int_0^\infty f(ut)e^{-t}dt, \quad u \in (-\tau_1, \tau_2).$$

Among other things, it was shown that the Sumudu transform has units-preserving features, and as a result, it may be utilised to address issues without turning to the frequency domain. As will be seen further below, this is one of the numerous advantages of this novel transform, particularly when applied to situations involving physical dimensions. In reality, the Sumudu transform, which is inherently linear, retains linear functions and, as a result, does not alter units, which is very important (see for instance Watugala [11] or Belgacem et al. [5]). On the surface, this statement is arguably best shown by the fact that it is an inference of a more general conclusion.

Theorem 5.2. The Sumudu transform amplifies the coefficients of the power series function,

$$f(t) = \sum_{n=0}^{\infty} a_n t^n,$$

by sending it to the power series function,

$$G(u) = \sum_{n=0}^{\infty} n! a_n u^n.$$

Proof. Let f (t) be in A. If f (t) = $\sum_{n=0}^{\infty} a_n t^n$ in some interval I ⊂ R, then by Taylor's function expansion theorem,

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n.$$

Therefore, by (5.2), and that of the gamma function Γ (see Table 2.1), we have

$$\mathbb{S}[f(t)] = \int_0^{\infty} \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} (ut)^n e^{-t} dt = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} u^n \int_0^{\infty} t^n e^{-t} dt$$
$$= \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} u^n \Gamma(n+1) = \sum_{k=0}^{\infty} f^{(n)}(0) u^n.$$

Consequently, it is perhaps worth noting that since

$$\mathbb{S}[(1+t)^m] = \mathbb{S} \sum_{n=0}^{m} C_n^m t^n = \mathbb{S} \sum_{n=0}^{m} \frac{m!}{n!(m-n)!} u^n = \sum_{n=0}^{m} \frac{m!}{(m-n)!} u^n = \sum_{n=0}^{m} P_n^m u^n,$$

Due to the fact that the Sumudu transform converts combinations (Cm n) into permutations (Pm n), it may seem that more order is introduced into discrete systems. In addition, when the following criteria are met, the criterion that S[f (t)] converges in an interval containing u = 0 is met, specifically, that the following conditions are met:

$$\text{(i) } f^{(n)}(0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$
$$\text{(ii) } \lim_{n \to \infty} \left| \frac{f^{(n+1)}(0)}{f^{(n)}(0)} u \right| < 1.$$

This means that the convergence radius r of S[f (t)] depends on the sequence f (n) (0), since

$$r = \lim_{n\to\infty} \left| \frac{f^{(n)}(0)}{f^{(n+1)}(0)} \right|.$$

Clearly, the Sumudu transform may be used as a signal processing or a detection tool, especially in situations where the original signal has a decreasing power tail. However,

It is necessary to use caution, particularly if the power series is not significantly decaying. This following case may serve as an educational illustration of the previously mentioned worry. To illustrate, consider the purpose of the word

$$f(t) = \begin{cases} \ln(t+1) & \text{if } t \in (-1,1], \\ 0 & \text{otherwise.} \end{cases}$$

Since $f(t) = \sum_{n=1}^{\infty} (-1)^{n-1}(t^n/n)$, then except for $u = 0$,

$$\mathbb{S}[f(t)] = \sum_{n=1}^{\infty} (-1)^{n-1}(n-1)!u^n$$

Diverges throughout, because its convergence radius

$$r = \lim_{n\to\infty} \left| \frac{(-1)^{n-1}(n-1)!}{(-1)^n n!} \right| = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Theorem 5.1 implies a transparent inverse transform in the discrete case, that of getting the original function from its given transform, in the obvious manner.

Up to null functions, the inverse discrete Sumudu transform, f (t), of the power series

$G(u) = \sum_{n=0}^{\infty} b_n u^n$, is given by

$$\mathbb{S}^{-1}[G(u)] = f(t) = \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) b_n t^n.$$

In the next section, we provide a general inverse transform formula, albeit in a complex setting.

5.3 EXISTENCE CONDITION FOR THE SUMUDU TRANSFORM

Theorem 5.3 If f is of exponential order, then its Sumudu transform S [f(t, x)] = F(v, u) exists and is given by

$$F(v,u) = \int_0^\infty \int_0^\infty e^{-\frac{t}{v}-\frac{x}{u}} f(t,x)\,dt\,dx,$$

where, $\frac{1}{u} = \frac{1}{\eta} + \frac{i}{\tau}$ *and* $\frac{1}{v} = \frac{1}{\mu} + \frac{i}{\xi}$. *The defining integral for* F *exists at points* $\frac{1}{u} + \frac{1}{v} = \frac{1}{\eta} + \frac{1}{\mu} + \frac{i}{\tau} + + \frac{i}{\xi}$ *in the right half plane* $\frac{1}{\eta} + \frac{1}{\mu} > \frac{1}{K_1} + \frac{1}{K_2}$.

Proof. Using $\frac{1}{u} = \frac{1}{\eta} + \frac{i}{\tau}$ and $\frac{1}{v} = \frac{1}{\mu} + \frac{i}{\xi}$, we can express F(v, u) as:

Proof Using $\frac{1}{u} = \frac{1}{\eta} + \frac{i}{\tau}$ and $\frac{1}{v} = \frac{1}{\mu} + \frac{i}{\xi}$, we can express F(v, u) as:

$$\begin{aligned} F(v,u) &= \int_0^\infty \int_0^\infty f(t,x)\cos\left(\frac{t}{\tau} + \frac{x}{\zeta}\right) e^{-\frac{t}{\eta}-\frac{x}{\mu}}\,dt\,dx \\ &\quad -i\int_0^\infty \int_0^\infty f(t,x)\sin\left(\frac{t}{\tau} + \frac{x}{\zeta}\right) e^{-\frac{t}{\eta}-\frac{x}{\mu}}\,dt\,dx. \end{aligned}$$

Then, for values of $\frac{1}{\eta} + \frac{1}{\mu} > \frac{1}{K_1} + \frac{1}{K_2}$, we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty |f(t,x)| \left|\cos\left(\frac{t}{\tau} + \frac{x}{\zeta}\right)\right| e^{-\frac{t}{\eta}-\frac{x}{\mu}}\,dt\,dx \\ \leq\ & M\int_0^\infty \int_0^\infty e^{\left(\frac{1}{K_1}-\frac{1}{\eta}\right)t+\left(\frac{1}{K_2}-\frac{1}{\mu}\right)x}\,dt\,dx \\ \leq\ & M\left(\frac{\eta K_1}{\eta - K_1}\right)\left(\frac{\eta K_2}{\mu - K_2}\right) \end{aligned}$$

And

$$\begin{aligned}
&\int_0^\infty \int_0^\infty |f(t,x)| \left|\sin\left(\frac{t}{\tau}+\frac{x}{\zeta}\right)\right| e^{-\frac{t}{\eta}-\frac{x}{\mu}} dt dx \\
\leq\ & M \int_0^\infty \int_0^\infty e^{\left(\frac{1}{K_1}-\frac{1}{\eta}\right)t+\left(\frac{1}{K_2}-\frac{1}{\mu}\right)x} dt dx \\
\leq\ & M\left(\frac{\eta K_1}{\eta - K_1}\right)\left(\frac{\eta K_2}{\mu - K_2}\right)
\end{aligned}$$

Which imply that the integrals defining the real and imaginary parts of F exist for value of Re $(\frac{1}{u}+\frac{1}{\mu}) > \frac{1}{K_1}+\frac{1}{K_2}$, and this completes the proof.

Consequently, we remark that the piecewise continuous and exponential order properties of a function f are necessary and sufficient criteria for the existence of the Sumudu transform for that function. Note also that the double Sumudu transform of function f(t, x) is defined in [5], and that it has the form

$$(2.1) \quad F(v,u) = S_2\left[f(t,x);(v,u)\right] = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-\left(\frac{t}{v}+\frac{x}{u}\right)} f(t,x) dt dx$$

Where, S2 indicates double Sumudu transform and f(t, x) is a function which can be expressed as a convergent infinite series. Now, it is well known that the derivative of convolution for two functions f and g is given by

$$\frac{d}{dx}(f*g)(x) = \frac{d}{dx}f(x)*g(x) \text{ or } f(x)*\frac{d}{dx}g(x)$$

And it can be easily proved that Sumudu transform is:

$$\begin{aligned}
S\left[\tfrac{d}{dx}(f*g)(x);\ v\right] = uS\left[\tfrac{d}{dx}f(x);u\right] S\left[g(x);\ u\right] \\
uS\left[f(x);\ u\right] S\left[\tfrac{d}{dx}g(x);\ u\right].
\end{aligned}$$

Strong correlations exist between the double Sumudu and double Laplace transforms, which may be stated in either of two ways:

$$\begin{aligned}
&\text{(I)} \quad uvF(u,v) = \pounds_2\left(f(x,y);\left(\tfrac{1}{u},\tfrac{1}{v}\right)\right) \\
&\text{(II)} \quad psF(p,s) = \pounds_2\left(f(x,y);\left(\tfrac{1}{p},\tfrac{1}{s}\right)\right)
\end{aligned}$$

In this case, £2 denotes the operation of the double Laplace transform. Most notably, the double Sumudu and double Laplace transformations swap the images of sin(x + t) and cos(x + t), respectively. It turns out that there is a

$$S_2\left[\sin(x+t)\right] = \mathcal{L}_2\left[\cos(x+t)\right] = \frac{u+v}{(1+u)^2(1+v)^2}$$

and $$S_2\left[\cos(x+t)\right] = \mathcal{L}_2\left[\sin(x+t)\right] = \frac{1}{(1+u)^2(1+v)^2}.$$

5.4 SOME BASIC PROPERTIES OF SUMUDU TRANSFORM

Basic Properties of the– Sumudu Transform

If $\overline{\phi}(\rho,\sigma) = L_xS_t[\phi(x,t)]$, then

$$(I)L_xS_t\left[\frac{\partial\phi(x,t)}{\partial x}\right] = \rho\overline{\phi}(\rho,\sigma) - S[\phi(0,t)],$$

$$(II)L_xS_t\left[\frac{\partial\phi(x,t)}{\partial t}\right] = \frac{1}{\sigma}\overline{\phi}(\rho,\sigma) - \frac{1}{\sigma}L(\phi(x,0)),$$

$$(III)L_xS_t\left[\frac{\partial^2\phi(x,t)}{\partial x^2}\right] = \rho^2\overline{\phi}(\rho,\sigma) - \rho S(\phi(0,t)) - S(\phi_x(0,t)),$$

$$(IV)L_xS_t\left[\frac{\partial^2\phi(x,t)}{\partial t^2}\right] = \frac{1}{\sigma^2}\overline{\phi}(\rho,\sigma) - \frac{1}{\sigma^2}L(\phi(x,0)) - \frac{1}{\sigma}L(\phi_t(x,0)),$$

$$(V)L_xS_t\left[\frac{\partial^2\phi(x,t)}{\partial x\partial t}\right] = \frac{\rho}{\sigma}\overline{\phi}(\rho,\sigma) - \frac{\rho}{\sigma}L(\phi(x,0)) - S(\phi_t(0,t)).$$

Proof

$$(I)L_xS_t\left[\frac{\partial\phi(x,t)}{\partial x}\right] = \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho x - t/\sigma}\frac{\partial\phi(x,t)}{\partial x}dxdt$$
$$= \frac{1}{\sigma}\int_0^\infty e^{-t/\sigma}dt\left\{\int_0^\infty e^{-\rho x}\frac{\partial\phi(x,t)}{\partial x}dx\right\}$$

Let $u = e^{-\rho x}, dv = (\partial\phi(x,t)/\partial x)\,dx$ thus

$$L_xS_t\left[\frac{\partial\phi(x,t)}{\partial x}\right]=\frac{1}{\sigma}\int_0^{\infty}e^{-t/\sigma}dt\left\{-\phi(0,t)+\rho\int_0^{\infty}e^{-\rho x}\phi(x,t)dx\right.$$
$$=\rho\overline{\phi}(\rho,\sigma)-S(\phi(0,t)).$$

$$(II)L_xS_t\left[\frac{\partial\phi(x,t)}{\partial t}\right]=\frac{1}{\sigma}\int_0^{\infty}\int_0^{\infty}e^{-\rho x-t/\sigma}\frac{\partial\phi(x,t)}{\partial t}dxdt$$
$$=\frac{1}{\sigma}\int_0^{\infty}e^{-\rho x}dx\left\{\int_0^{\infty}e^{-t/\sigma}\frac{\partial\phi(x,t)}{\partial t}dt\right\}$$

let $u=e^{-t/\sigma}$, $dv=\partial\phi(x,t)/\partial t\ dt$, then

$$L_xS_t\left[\frac{\partial\phi(x,t)}{\partial t}\right]=\frac{1}{\sigma}\int_0^{\infty}e^{-\rho x}dx\left\{-\phi(x,0)+\frac{1}{\sigma}\int_0^{\infty}e^{-t/\sigma}\phi(x,t)dt\right.$$
$$=\frac{1}{\sigma}\overline{\phi}(\rho,\sigma)-\frac{1}{\sigma}L(\phi(x,0)).$$

Similarly, we can prove that (III), (IV), and (V).

5.5 THE CONVOLUTION THEOREM OF SUMUDU TRANFORM

Through the course of this chapter, the symbols for the sets of all complex numbers, the set of all real numbers, the set of all integers, the set of all natural numbers, and the set of all non-negative integers will be used to refer to the sets of all real numbers, the set of all real numbers (including zero), the set of all real numbers (including zero), the set of all non-negative integers (including zero), and the set of all non-negative integers (including zero). Integral transformations have been shown to be quite useful in the solution of differential and integrodifferential problems (see [1-12]). The Laplace transform is one of the most effective integral transforms, where f is a function specified for t 0, defined by the formula

$$F(s)=(f(t))=\int_0^{\infty}e^{-st}f(t)\,dt,$$

Assuming, of course, that the integral converges cites several examples of how it has tremendous applications not just in applied mathematics but also in other disciplines of science like as astronomy, engineering, physics, and so on. In addition, several integral transforms such as the Sumudu, Fourier, Elzaki, and M-transforms have been addressed, and their characteristics and applications have been thoroughly investigated by a large number of scientists, see and the references listed therein for more information. This is the theoretical equivalent of the Sumudu transform, which was first developed and is given by the formula

$$G(u) = \mathbf{S}[f(t)] = \int_0^\infty e^{-t} f(ut)\,dt = \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}} f(t)\,dt, \quad u \in (-\tau_1, \tau_2),$$

Over the set of functions

$$A = \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}.$$

There have been several investigations and studies of the Sumudu transform by physicists and mathematicians, as seen in and other publications. For example, described the Sumudu transform as a two-variable transformation and offered an example of how to solve partial differential equations with known beginning conditions. obtained the Sumudu transform of partial derivatives and illustrated the usefulness of the transform by applying three distinct partial differential equations in his research. Several properties of the Sumudu transform, as well as the relationship between the Sumudu and Laplace transforms, were investigated by Kilicman et al. [6], who then presented an application of the double Sumudu transform to solve the wave equation in one dimension that exhibits singularity at the initial conditions. According to Asiru a Sumudu transform of numerous special functions was supplied, and several applications were obtained using Abel's integral equation, an integrodifferential equation, a dynamic system with delayed time signals, and a differential dynamic system, among others.

Belgacem and colleagues discovered key features of the Sumudu transform, including scale and unit-preserving qualities, and shown that the transform may be used to solve an integral production-depreciation issue. Belgacem went into further detail on the features and connections of Sumudu. All current Sumudu integration, differentiation, and Sumudu shifting theorems as well as convolution theorems were extended by Belgacem and colleagues Here, we present the modified Sumudu transform and analyse a wide range of features and relationships, including the power function, the sine and cosine of a given power function, hyperbolic sine and hyperbolic cosine of a given power function, and function derivatives of a given function. In addition, we find two shifting characteristics for the modified Sumudu transform as well as a scale preserving theorem for it. This chapter presents a modified inverse Sumudu transform and deduces several relationships and instances from it. Also shown is that modified Sumudu transform is the theoretical counterpart transform of modified Laplace transform, as seen in the figure below. At the end of this section, we discuss duality between the modified Laplace transformation and the modified Sumudu transformation.

The Sumudu transformation satisfies the following operational properties, cf.:

$\mathbf{S}[1] = 1$	$\mathbf{S}[\sin(at)] = \frac{au}{1+u^2a^2}$
$\mathbf{S}[t] = u$	$\mathbf{S}[\cos(at)] = \frac{1}{1+u^2a^2}$
$\mathbf{S}[t^n] = n!u^n$	$\mathbf{S}[\sinh(at)] = \frac{au}{1+u^2a^2}$
$\mathbf{S}[e^{at}] = \frac{1}{1-ua}$	$\mathbf{S}[\cosh(at)] = \frac{1}{1+u^2a^2}$
$\mathbf{S}[f(at)] = G(au)$	$\mathbf{S}[e^{at}f(t)] = \frac{1}{1-au}G\left(\frac{u}{1-au}\right)$

Let f (t),g (t) ∈ A be Sumudu transforms M (u) and N (u), respectively. Then the Sumudu transform of the convolution of f and g is given by

$$\mathbf{S}[(f * g)(t)] = uM(u)N(u),$$

Where the convolution integral is given by (cf. [2,4])

$$(f * g)(t) = \int_0^t g(x) f(t-x)\, dx$$

Where s is a complex number with Re (s) > 0 The gamma function satisfies the following relations

$$\Gamma(s+1) = s\Gamma(s) \text{ and } \Gamma(n+1) = n!$$

For n being a non-negative integer.

5.6 DIFFERENTIATION AND INTEGRATION OF SUMUDU TRANSFORMS

This following theorem was established by Belgacem et al. [5] using the LSD between the Sumudu transform and the Laplace transform. While we mention it in order to keep this work self-contained, we utilise an induction argument to support the conclusion here.

Theorem 5.6.1 Let f (t) be in A, and let Gn(u) denote the Sumudu transform of the nth derivative, f (n) (t) of f (t), then for n 1,

$$G_n(u) = \frac{G(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-k}}.$$

Proof For n = 1, shows that holds. To proceed to the induction step, we assume that holds for n and prove that it carries to n+ 1. Once more by virtue of we have

$$G_{n+1}(u) = \mathbb{S}[(f^{(n)}(t))'] = \frac{\mathbb{S}\left[f^{(n)}(t)\right] - f^{(n)}(0)}{u}$$
$$= \frac{G_n(u) - f^{(n)}(0)}{u} = \frac{G(u)}{u^{n+1}} - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{u^{n+1-k}}.$$

In particular, this means that the Sumudu transform, G2(u), of the second derivative of the function, f (t), is given by

$$G_2(u) = \mathbb{S}(f''(t)) = \frac{G(u) - f(0)}{u^2} - \frac{f'(0)}{u}.$$

For instance, the general solution of the second-order equation,

$$\frac{d^2 y(t)}{dt^2} + w^2 y(t) = 0,$$

Can easily be transformed into its Sumudu equivalent,

$$\frac{G(u) - y(0)}{u^2} - \frac{y'(0)}{u} - w^2 G(u) = 0,$$

With general Sumudu solution,

$$G(u) = \frac{y(0) + uy'(0)}{1 + w^2 u^2},$$

And upon inverting, by using Theorem 5.1 (or see Table 5.1), we get the general time solution:

$$y(t) = y(0)\cos(wt) + \frac{y'(0)}{w}\sin(wt).$$

Obviously, Theorem 5.1 shows that the Sumudu transform can be used just like the Laplace transform, as in the previous example, to solve both linear differential equations of any order. The next theorem allows us to use the Sumudu transform as efficiently to solve differential equations involving multiple integrals of the dependent variable as well, by rendering them into algebraic ones.

Theorem 5.6.2 Let f (t) be in A, and let Gn(u) denote the Sumudu transform of the nth antiderivative of f (t), obtained by integrating the function, f (t), n times successively

$$W^n(t) = \iint_0^t \cdots \int_0^t f(\tau)(d\tau)^n,$$

Then for n 1,

$$G^n(u) = \mathbb{S}(W^n(t)) = u^n G(u).$$

Proof. For n = 1, (3.6) shows that (4.9) holds. To proceed to the induction step, we assume that (4.9) holds for n, and prove it carries to n + 1. Once more, by virtue of (3.6), we have

$$G^{n+1}(u) = \mathbb{S}(W^{n+1}(t)) = \mathbb{S}\left[\int_0^t W^n(\tau)d\tau\right] = u\mathbb{S}[W^n(t)] = u[u^n G(u)] = u^{n+1}G(u). \quad (4.$$

This theorem generalizes the Sumudu convolution Theorem 4.1 in Belgacem et al. [5], which states that the transform of

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau,$$

Is given by

$$\mathbb{S}((f * g)(t)) = uF(u)G(u).$$

Let f (t),g(t),h(t),h1(t),h2(t),..., and hn(t) be functions in A, having Sumudu transforms, F(u),G(u),H(u),H1(u),H2(u),..., and Hn(t), respectively, then the Sumudu transform of

$$(f * g)^n(t) = \iint_0^t \cdots \int_0^t f(\tau)g(t-\tau)(d\tau)^n$$

is given by

$$\mathbb{S}((f * g)^n(t)) = u^n F(u)G(u).$$

Moreover, for any integer n 1,

$$\mathbb{S}[(h_1 * h_2 * \cdots * h_n)(t)] = u^{n-1} H_1(u)H_2(u)\cdots H_n(u).$$

In particular, the Sumudu transform of (f ∗ g ∗ h), with f , g, h in A, is given by

$$\mathbb{S}[(f * g * h)(t)] = u^2 F(u)G(u)H(u).$$

Proof. Equation (4.14) is just a straightforward consequence of Theorem 4.2, owing to the property of associativity of the convolution operator, (4.12), which indicates that the convolution operator is associative (4.15). After everything is said and done, (4.16) is just an implication of (4.15) with n = 3. The preceding conclusions may be used in a strong way to the solution of integral, differential, and integrodifferential equations, among others. Applications of this kind are discussed in the literature review portion of the article introduction section, which includes the cited sources. To emphasise this point, we would like to remind the reader that Belgacem et al. [5] employed similar findings to solve convolution type equations.

5.7 DIVISION OF SUMUDU TRANSFORMS

Over the set of functions,

$$A = \{f(t) \mid \exists M, \tau_1, \tau_2 > 0, \mid f(t) \mid < Me^{|t|/\tau_j}, \text{ if } t \in (-1)^j \times [0, \infty)\},$$

The Sumudu transform is defined by

$$G(u) = \mathbb{S}[f(t)] = \int_0^\infty f(ut)e^{-t}dt, \quad u \in (-\tau_1, \tau_2).$$

Among other things, it was shown that the Sumudu transform has units-preserving features, and as a result, it may be utilised to address issues without turning to the frequency domain. As will be seen further below, this is one of the numerous advantages of this novel transform, particularly when applied to situations involving physical dimensions. In reality, the Sumudu transform, which is inherently linear, retains linear functions and, as a result, does not alter units, which is very important (see for instance Watugala [11] or Belgacem et al. [5]). On the surface, this statement is arguably best shown by the fact that it is an inference of a more general conclusion.

Theorem 2.1. The Sumudu transform amplifies the coefficients of the power series function,

$$f(t) = \sum_{n=0}^{\infty} a_n t^n,$$

by sending it to the power series function,

$$G(u) = \sum_{n=0}^{\infty} n! a_n u^n.$$

Proof Let f (t) be in A. If f (t) = $\sum_{n=0}^{\infty} a_n t^n$ in some interval I ⊂ R, then by Taylor's function expansion theorem,

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n.$$

Therefore, by (2.2), and that of the gamma function Γ (see Table 5.1), we have

$$\mathbb{S}[f(t)] = \int_0^{\infty} \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} (ut)^n e^{-t} dt = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} u^n \int_0^{\infty} t^n e^{-t} dt$$
$$= \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} u^n \Gamma(n+1) = \sum_{k=0}^{\infty} f^{(n)}(0) u^n.$$

Consequently, it is perhaps worth noting that since

$$\mathbb{S}[(1+t)^m] = \mathbb{S} \sum_{n=0}^{m} C_n^m t^n = \mathbb{S} \sum_{n=0}^{m} \frac{m!}{n!(m-n)!} u^n = \sum_{n=0}^{m} \frac{m!}{(m-n)!} u^n = \sum_{n=0}^{m} P_n^m u^n,$$

Due to the fact that the Sumudu transform converts combinations (Cm n) into permutations (Pm n), it may seem that more order is introduced into discrete systems. In addition, when the following criteria are met, the criterion that S[f (t)] converges in an interval containing u = 0 is met, specifically, that the following conditions are met:

(i) $f^{(n)}(0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$

(ii) $\lim_{n\to\infty}\left|\dfrac{f^{(n+1)}(0)}{f^{(n)}(0)}u\right| < 1.$

This means that the convergence radius r of S[f (t)] depends on the sequence f (n) (0), since

$$r = \lim_{n\to\infty}\left|\frac{f^{(n)}(0)}{f^{(n+1)}(0)}\right|.$$

Clearly, the Sumudu transform may be used as a signal processing or a detection tool, especially in situations where the original signal has a decreasing power tail.

5.8 SUMUDU TRANSFORM OF SPECIAL FUNCTIONS

Table 5.1 Special functions

(1) Gamma function	$\Gamma(n) = \int_0^\infty u^{n-1}e^{-u}du,\ n > 0$
(2) Beta function	$B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}du = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$
(3) Bessel function	$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)} \times \left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \cdots\right\}$
(4) Modified Bessel function	$I_n(x) = i^{-n}J_n(ix) = \frac{x^n}{2^n\Gamma(n+1)} \times \left\{1 + \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \cdots\right\}$
(5) Error function	$\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}du$
(6) Complementary error function	$\mathrm{erf}(t) = 1 - \mathrm{erf}(t) = \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-u^2}du$
(7) Exponential integral	$\mathrm{Ei}(t) = \int_t^\infty \frac{e^{-u}}{u}du$
(8) Sine integral	$\mathrm{Si}(t) = \int_0^t \frac{\sin u}{u}du$
(9) Cosine integral	$\mathrm{Ci}(t) = \int_0^t \frac{\cos u}{u}du$
(10) Fresnel sine integral	$S(t) = \int_0^t \sin u^2 du$
(11) Fresnel cosine integral	$C(t) = \int_0^t \cos u^2 du$
(12) Laguerre polynomials	$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}),\quad n = 0, 1, 2, \ldots$

It is necessary to use caution, particularly if the power series is not significantly decaying. This following case may serve as an educational illustration of the previously mentioned worry. To illustrate, consider the purpose of the word

$$f(t) = \begin{cases} \ln(t+1) & \text{if } t \in (-1, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Since $f(t) = \sum_{n=1}^{\infty}(-1)^{n-1}(t^n/n)$, then except for u = 0,

$$\mathbb{S}[f(t)] = \sum_{n=1}^{\infty}(-1)^{n-1}(n-1)!u^n$$

Diverges throughout, because its convergence radius

$$r = \lim_{n\to\infty}\left|\frac{(-1)^{n-1}(n-1)!}{(-1)^n n!}\right| = \lim_{n\to\infty}\frac{1}{n} = 0.$$

a transparent inverse transform in the discrete case, i.e., the ability to get the original function from its given transform in a clear way

Up to null functions, the inverse discrete Sumudu transform, f (t), of the power series G(u) = ∞ n=0 bnun, is given by

$$\mathbb{S}^{-1}[G(u)] = f(t) = \sum_{n=0}^{\infty}\left(\frac{1}{n!}\right)b_n t^n.$$

In the next section, we provide a general inverse transform formula, albeit in a complex setting

5.9 THE RELATION BETWEEN SUMUDU TRANSFORM AND LAPLACE TRANSFORM

The single Sumudu transform is defined over the set

$$A = \{f(t)|\exists M,\ \tau_1,\tau_2>0,\ |f(t)|<Me^{t/\tau_j}\ \text{if}\ t\in(-1)^j\times[0,\infty)\}$$

By

$$F(u) = S[f(t);u] = \frac{1}{u}\int_0^{\infty} e^{-(t/u)}f(t)\,dt, \quad u\in(-\tau_1,\tau_2)$$

See [8], and the double Sumudu transform of function $f(t,x); t,x \in \mathbb{R}_+$, is defined by

$$F(v,u) = S_2[f(t,x);(v,u)] = \frac{1}{uv}\int_0^{\infty}\int_0^{\infty} e^{-(t/v+x/u)}f(t,x)\,dt\,dx$$

Given that an integral exists in which S2 denotes double Sumudu transform and f(t,x) denotes a function that can be represented as a convergent infinite series, the condition is satisfied. The Sumudu transform of derivative for convolution is then introduced as follows: the convolution of derivative of the two functions f(x) and g(x) is provided by the sumudu transform of derivative for convolution

$$\frac{d}{dx}(f*g)(x) = \frac{d}{dx}f(x)*g(x) \quad \text{or} \quad f(x)*\frac{d}{dx}g(x)$$

And the Sumudu transform is given by

$$S\left[\frac{d}{dx}(f*g)(x);v\right] = uS\left[\frac{d}{dx}f(x);u\right]S[g(x);u] \quad \text{or} \quad uS[f(x);u]S\left[\frac{d}{dx}g(x);u\right].$$

The convolution theorem holds true for a variety of integral transforms, including the Laplace transform, the two-sided Laplace transform, and, with appropriate modification, the Mellin transform and the Hartley transform among others. This theorem holds true for the Sumudu transform as well, as seen below.

Theorem 1. Let f(t,x) and g(t,x) be having double Sumudu transform. Then double Sumudu transform of the double convolution of the f(t,x) and g(t,x),

$$(f**g)(t,x) = \int_0^t \int_0^x f(\zeta,\eta)g(t-\zeta,x-\eta)\,d\zeta\,d\eta$$

Is given by

$$S_2[(f**g)(t,x);v,u] = uvF(v,u)G(v,u).$$

Proof By using the definition of double Sumudu transform and double convolutions, we have

$$\begin{aligned} S_2[(f**g)(t,x);u,v] &= \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-(t/u+x/v)}(f**g)(t,x)\,dt\,dx \\ &= \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-(t/u+x/v)}\left(\int_0^t \int_0^x f(\zeta,\eta)g(t-\zeta,x-\eta)\,d\zeta d\eta\right)dt\,dx \end{aligned}$$

Let $\alpha = t-\zeta$ b ¼ xZ and using the valid extension of the upper bound of integrals to t-1 and x-1, we yield

$$S_2[(f_**g)(t,x);u,v] = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-(\zeta/u-\eta/v)}f(t-\alpha,x-\beta)\,d\zeta\,d\eta \int_{-\zeta}^\infty \int_{-\eta}^\infty e^{-(\alpha/u-\beta/v)}g(\alpha,\beta)\,d\alpha\,d\beta$$

Both functions f(t,x) and g(t,x) have zero value for to0, and xo0, thus it follows with respect to the lower limit of integrations that

$$S_2[(f_{*}*g)(t,x);u,v] = \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-(\zeta/v-\eta/u)}f(\zeta,\eta)\,d\zeta\,d\eta\int_0^\infty\int_0^\infty e^{-(\alpha/v-\beta/u)}g(\alpha,\beta)\,d\alpha\,d\beta$$

Then, it is easy to see that

$$S_2[(f**g)(t,x);u,v] = uvF(u,v)G(u,v)$$

And the double Sumudu transform of derivative of the double convolution given by

$$S_2\left[\frac{\partial}{\partial x}(f**g)(t,x);v,u\right] = uvS_2\left[\frac{\partial}{\partial x}f(t,x);v,u\right]S_2[g(t,x);v,u]$$
$$\text{or}\quad uvS_2[f(t,x);v,u]S_2\left[\frac{\partial}{\partial x}g(t,x);v,u\right]. \qquad \square$$

The triple Sumudu and Laplace transforms having strong relation that may be expressed as

$$uvwF(u,v,w) = £_3\left(f(x,y,t);\left(\frac{1}{u},\frac{1}{v},\frac{1}{w}\right)\right),$$

Where £3 represents the operation of triple Laplace transform In particular, this relation is best illustrated by the fact that the triple Sumudu and triple Laplace transforms interchange the image of sin (xþyþt) and cos(xþyþt). It turns out that

$$S_3[\sin(x+y+t)] = £_3[\cos(x+y+t)] = \frac{u+v+w}{(1+u)^2(1+v)^2(1+w)^2}$$

And

$$S_3[\cos(x+y+t)] = £_3[\sin(x+y+t)] = \frac{1}{(1+u)^2(1+v)^2(1+w)^2}.$$

Thus, in particular case the relation between double Sumudu of convolution and double Laplace transform of convolution is given by

$$S_2[(f**g)(t,x);v,u] = \frac{1}{uv}£_2(f**g)(t,x).$$

In the following theorem, we discuss the triple Sumudu transform of periodic function f as follows.

5.10 THE INVERSE OF LAPLACE TRANSFORM AND EXAMPLES

Recall the solution procedure outlined in Figure 6.1. The final stage in that solution procedure involves calulating inverse Laplace transforms. In this section we look at the problem of finding inverse Laplace transforms. In other words, given F(s), how do we find f(x) so that F(s) = L[f(x)].

We begin with a simple example which illustrates a small problem on finding inverse Laplace transforms.

Example: Consider the functions

$$f(x) = x^2, \qquad \text{and} \qquad g(x) = \begin{cases} x^2 & x \neq 2,3 \\ 48 & x = 2 \\ -\pi & x = 3 \end{cases} .$$

Then $\mathfrak{L}[f(x)] = \mathfrak{L}[g(x)] = \frac{2}{s^3}$. Because the altering of an integral's integrand at a few isolated locations has no effect on the integral, it is possible for more than one function to have the same Laplace transform. Example 6.24 demonstrates that inverse Laplace transforms are not one-to-one correspondences. But it is possible to demonstrate that when numerous functions have the same Laplace transform, at most one of them is continuous, as demonstrated in the following example. As a result, we have come up with the following definition.

Definition: The inverse Laplace transform of F(s), denoted $\mathfrak{L}^{-1}[F(s)]$, is the function f defined on [0,∞) which has the fewest number of discontinuities and satisfies

$$\mathfrak{L}[f(x)] = F(s).$$

Example

1. $\mathfrak{L}^{-1}[\frac{2}{s^3}] = x^2.$
2. $\mathfrak{L}^{-1}[\frac{s}{s^2+9}] = \cos 3x.$
3. $\mathfrak{L}^{-1}[\frac{s-1}{s^2-2s+5}] = \mathfrak{L}^{-1}[\frac{s-1}{(s-1)^2+4}] = e^x\,\mathfrak{L}^{-1}[\frac{s}{s^2+4}] = e^x \cos 2x.$ (using property 1 of Theorem 6.17 in reverse) ∎

The inverse Laplace transform is a linear operator.

Theorem:

If $\mathfrak{L}^{-1}[F(s)]$ *and* $\mathfrak{L}^{-1}[G(s)]$ *exist, then* $\mathfrak{L}^{-1}[\alpha F(s)+\beta G(s)] = \alpha\mathfrak{L}^{-1}[F(s)] + \beta\mathfrak{L}^{-1}[G(s)]$.

Proof

Starting from the right hand side we have

$$\mathfrak{L}[\alpha\mathfrak{L}^{-1}[F(s)] + \beta\mathfrak{L}^{-1}[G(s)]] = \alpha\mathfrak{L}[\mathfrak{L}^{-1}[F(s)]] + \beta\mathfrak{L}[\mathfrak{L}^{-1}[G(s)]] = \alpha F(s) + \beta G(s).$$

The result follows.

Most of the properties of the Laplace transform can be reversed for the inverse Laplace transform.

Theorem

If $\mathfrak{L}^{-1}[F(s)] = f(x)$, *then the following hold:*

1. $\mathfrak{L}^{-1}[F(s+a)] = e^{-ax}f(x)$;
2. $\mathfrak{L}^{-1}[sF(s)] = f'(x)$, *if* $f(0)=0$;
3. $\mathfrak{L}^{-1}[\frac{1}{s}F(s)] = \int_0^x f(t)\,dt$;
4. $\mathfrak{L}^{-1}[e^{-as}F(s)] = u_a(x)\,f(x-a)$.

Proof

1. $\mathfrak{L}[e^{-ax}f(x)] = F(s+a)$ from Theorem 6.17, property 1. The result follows.
2. $\mathfrak{L}[f'(x))] = -f(0) + sF(s)$ from Theorem 6.17, property 4. The result follows.
3. $\mathfrak{L}[\int_0^x f(t)\,dt] = \frac{1}{s}F(s)$, from Theorem 6.17, property 5. The result follows.
4. $\mathfrak{L}[u_a(x)f(x-a)] = e^{-as}\mathfrak{L}[f(x)] = e^{-as}F(s)$, from Theorem 6.19. The result follows.

Example:

. Find $\mathfrak{L}^{-1}[\frac{1}{s(s^2+1)}]$.

Solution

We can write $\frac{1}{s(s^2+1)} = \frac{1}{s}F(s)$, where $F(s) = \frac{1}{s^2+1}$. Then $f(x) = \mathfrak{L}^{-1}[F(s)] = \sin x$, so we get

$$\mathfrak{L}^{-1}[\frac{1}{s(s^2+1)}] = \mathfrak{L}^{-1}[\frac{1}{s}F(s)] = \int_0^x f(t)\,dt = \int_0^x \sin t\,dt = 1 - \cos x. \quad \blacksquare$$

5.11 THE PROPERTIES OF INVERSE SUMUDU TRANSFORMAND EXAMPLES

The Sumudu transform has already shown great promise, owing to its straightforward construction and the unusual and valuable qualities that result as a result. It has been shown here and elsewhere that it may be used to assist in the solution of complex issues in engineering mathematics and applied science. Despite the promise offered by this new operator, only a few theoretical studies have been published in the literature during a fifteen-year period, despite the possibility offered by this new operator. The Sumudu transform is not mentioned in most, if not all, of the transform theory texts that are currently accessible. No mention of the Sumudu transform can be found in any of the more recent well-known comprehensive handbooks, including.

Perhaps this is due to the fact that no transform with this name (in the traditional sense) was announced until the late 1980s and early 1990s of the previous century. On the other hand, it is important to note that an analogous formulation, known as the s-multiplied Laplace transform, was disclosed as early as 1948 (see, for example, and references), if not before. The publication of which demonstrated the application of the Sumudu transform to partial differential equations, came soon after Watugala's key work .

It was shown in work that the Sumudu transform may be utilised to solve ordinary differential equations and engineering control problems with high accuracy. Watugala's work was followed who introduced a complicated inversion formula for the Sumudu transform, which was followed by other researchers (see Theorem 5.1 in Section 5). The relatively recent series demonstrated how to use the transform to solve integrodifferential equations, with a focus on dynamic systems, and how to utilise the transform to solve differential equations. expanded the transform to two variables, with a focus on partial differential equations and solutions to these equations. demonstrated applications of convolution type integral equations, with a particular emphasis on difficulties in the manufacturing industry. This novel transform's core helpful qualities were established or corroborated by using a Laplace-Sumudu duality, which was noted in the previous section. An example of this would be a two-page table including the transforms of some of the fundamental functions. We give Sumudu transformations for a more thorough list of functions at the conclusion of this study, which is equivalent to the large Laplace transform list contained in Spiegel [9]. The introduction of more broad shift theorems, which seem to have combinatorial linkages to generalised stirling numbers, is in addition to a paradigm shift in the conceptual process of Sumudu transform use with regard to applications to differential equations. In addition, we prove more broad Sumudu differentiation, integration, and convolution theorems than have previously been proved in the literature, which is a significant advance. We also use the Laplace-Sumudu duality (LSD) to generate a complex inverse Sumudu transform, which is expressed as a formula for the Bromwhich contour integral.

CHAPTER 6

APPLICATIONS OF SUMUDU TRANSFORMS

6.1 INTRODUCTION

Through the course of this chapter, the symbols C (for complex numbers), R (for real numbers), N (for non-negative integers), Z (for real numbers), and N0 (for non-negative integers) will refer to the sets of all complex numbers, real numbers, integers, natural numbers, and non-negative integers, respectively.

Integral transformations have been shown to be quite useful in the solution of differential and integrodifferential problems (see). The Laplace transform is one of the most effective integral transforms, where f is a function specified for t 0, defined by the formula

$$F(s) = (f(t)) = \int_0^\infty e^{-st} f(t)\, dt,$$

Supposing that the integral is converging in some way cites several examples of how it has tremendous applications not just in applied mathematics but also in other disciplines of science like as astronomy, engineering, physics, and so on. In addition, several integral transforms such as the Sumudu, Fourier, Elzaki, and M-transforms have been addressed, and their characteristics and applications have been thoroughly investigated by a large number of scientists, see and the references listed therein for more information. This is the theoretical equivalent of the Sumudu transform, which was first developed and is given by the formula

$$G(u) = \mathbf{S}[f(t)] = \int_0^\infty e^{-t} f(ut)\, dt = \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}} f(t)\, dt, \quad u \in (-\tau_1, \tau_2),$$

Over the set of functions

$$A = \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}.$$

Several applications of the Sumudu transform have been examined and studied by a large number of physicists and mathematicians, as seen in the following examples: Examples include defining two variables Sumudu transform and providing an example of solving partial differential equations with known beginning conditions, as described by Watagula. Weerakoon obtained the Sumudu transform of partial derivatives and illustrated the usefulness of the transform by applying three distinct partial differential equations in his research. Following a brief discussion of various aspects of the Sumudu transform and relationships between the Sumudu and Laplace transforms, an application of the double Sumudu transform was presented, which was used to solve the wave equation in one dimension with singularity at beginning conditions. According to a Sumudu transform of numerous special functions was supplied, and several applications were obtained using Abel's integral equation, an integrodifferential equation, a dynamic system with delayed time signals, and a differential dynamic system, among others. Belgacem and colleagues discovered key features of the Sumudu transform, including scale and unit-preserving qualities, and shown that the transform may be used to solve an integral production-depreciation issue.

went into further detail on the features and connections of Sumudu. Sumudu integration, differentiation, and Sumudu shifting theorems, as well as convolution theorems, were all extended in this work. Here, we present the modified Sumudu transform and analyse a wide range of features and relationships, including the power function, the sine and cosine of a given power function, hyperbolic sine and hyperbolic cosine of a given power function, and function derivatives of a given function. In addition, we find two shifting characteristics for the modified Sumudu transform as well as a scale preserving theorem for it.

6.2 SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

The Sumudu transform was first introduced by Watugala (1993) for the purpose of solving differential equations and control engineering issues in control engineering. The Sumudu transform was used for functions of two variables in Watugula (2002), and the results were positive. Weerakoon was the founder of several of the assets under question (1994, 1998). Further essential features of this transform were discovered by Asiru (2002), who also verified their existence. In a similar vein, the one-dimensional neutron transport equation in Kadem was transformed using this technique (2005). The link between the Sumudu integral transform and other integral transforms has been shown (see Kilicman et al. for more information) (2011). It was Kilicman who established the relationship between the Sumudu transform and the Laplace transform, in particular (2011). Kilicman and Eltayeb (2010) investigated the characteristics of distributions using the Sumudu transform, which was further developed in Eltayeb et al. (2010). Kilicman and Eltayeb (2010) investigated the properties of distributions using the Sumudu transform (2010). Recently, this transform has been used to the solution of a system of differential equations (see Kilicman et al. for more information) (2010). It should be noted that one particularly intriguing point regarding the Sumudu transform is that the Original function and its Sumudu transform have the identical Taylor coefficients, with the exception of the factor.

n, See Zhang (2007). Thus if $f(t) = \sum_{n=-\infty}^{\infty} a_n t^n$ Then $F(u) = \sum_{n=-\infty}^{\infty} n!\, a_n u^n$, see Kilicman et al. (2011). Similarly, the Sumudu transform sends combinations $C(m,n)$ into permutations, $P(m,n)$ and hence it will be useful in the discrete systems. The Sumudu transform is defined by the formula

$$F(u) = s[f(t);u]$$

$$= \frac{1}{u}\int_{-\infty}^{\infty} e^{-\frac{t}{u}} f(t)\,dt\,,$$

$$u \epsilon (-\tau_1, \tau_2).$$

Over the set of

$$= \left\{ f(t) \Big|_{if\ t\epsilon(-1)_j X(-\infty,\infty)}^{\exists M, and\ or/, \tau_2 > 0, such\ that\ |f(t)| < Me^{\frac{|t|}{\tau_j}}} \right\}$$

We want to demonstrate the usefulness of this intriguing novel transform, as well as its efficiency, in solving linear ordinary differential equations with constant and non constant coefficients that include the non homogeneous component as convolutions, as part of this research.

6.3 PARTIAL DIFFERENTIAL EQUATIONS, INITIAL AND BOUNDARY VALUE PROBLEMS

In this section we introduce the notion of a partial differential equation and illustrate it with various examples.

6.3.1 What is a partial differential equation?

A partial differential equation (PDE) is a mathematical equation that describes the relationship between a function u of many variables Xl,"" xn and its partial derivatives from a strictly mathematical standpoint. A differential equation with more than one variable is differentiated from an ordinary differential equation, which only applies to functions of a single variable. For example, if a function of two variables is given by the notation u(x, y), then one may consider the following examples of partial differential equations as examples of functions of two variables:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(Laplace's equation)}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(the wave equation)}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0 \qquad \text{(the heat equation)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x, y) \qquad \text{(Poisson's equation)}$$

In order to simplify the notation, we will often use subscripts to denote the various partial derivatives, so that Ux = au/ax, Uxx = a2u/ax2 , and so forth. In this notation, the above four examples are written, respectively,

$$u_{xx} + u_{yy} = 0, \qquad u_{xx} - u_{yy} = 0, \qquad u_{xx} - u_y = 0, \qquad u_{xx} + u_{yy} = g$$

The order of a PDE is indicated by the highest-order derivative that appears. All of the above four examples are PDEs of second order.

In the case of a function of several variables $u(x_1, \ldots, x_n)$ the most general second-order partial differential equation can be written

$$F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1x_1}, u_{x_1x_2}, \ldots, u_{x_nx_n}) = 0$$

where the dots suggest the additional partial derivatives that may arise. In case n = 1 we get the second-order ordinary differential equation F(x, '1.£, '1.£', '1.£11) = 0. The essential knowledge on ordinary differential equations is reviewed in Appendix A.I. Another key idea linked to a PDE is that of linearity. This is most readily understood in the context of a differential operator

$ applied to a function u. Examples of differential operators include £'1.£ = au/ax, £'1.£ = 3u + siny8u/8x, and £'1.£ = '1.£ {)2u/8x2 • The operator is considered to be linear if for any two functions u, v and any constant c,

$$\mathcal{L}(u+v) = \mathcal{L}u + \mathcal{L}v, \qquad \mathcal{L}(cu) = c\mathcal{L}u$$

A PDE is said to be linear if it can be written in the form

(0.1.1) £u=g

(0.1.2)

Where £ is a linear differential operator and 9 is a given function. In case 9 = 0, (0.1.1) is said to be homogeneo'U..'J. For example, three of the above examples (Laplace's equation, the wave equation, and the heat equation) are linear homogeneous PDEs. The most general linear second-order PDE in two variables is written

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y)$$

Where the functions a, b, c, d, e, I, 9 are given.

6.4 SOLUTION OF INTEGRAL EQUATIONS

When a function to be determined occurs beneath the integral sign, the equation is said to be an integral equation. The linear integral equation in its most generic version is denoted by

$$h(x)\, u(x) = f(x) + \int_a^{b(x)} K(x\,,\ \xi)\, u(\xi)\, d\xi \quad \text{for all } x \in [a, b]$$

In which, u(x) is the function to be determined and K(x, ξ) is called the Kernel of integral equation.

6.4.1 Volterra Integral equation

$$h(x)\, u(x) = f(x) + \int_a^{x} K(x\,,\ \xi)\, u(\xi)\, d\xi \quad \text{for all } x \in [a, b]$$

That is, in Volterra equation b(x) = x

(i) If h(x) = 0, the above equation reduces to

$$-f(x) = \int_a^x K(x\,,\ \xi)\, u(\xi)\, d\xi$$

This equation is called Volterra integral equation of first kind.

(ii) If h(x) = 1, the above equation reduces to

$$u(x) = f(x) + \int_a^x K(x\,,\ \xi)\, u(\xi)\, d\xi$$

This equation is called Volterra integral equation of second kind.

6.4.2 Homogeneous integral equation. If f(x) = 0 for all x ∈ [a, b], then the reduced equation

$$h(x)\, u(x) = \int_a^{bx} K(x\,,\ \xi)\, u(\xi)\, d\xi$$

Is called homogeneous integral equation. Otherwise, it is called non-homogeneous integral equation.

6.4.3 Leibnitz Rule. The Leibnitz rule for differentiation under integral sign:

$$\frac{d}{dx}\left[\int_{\alpha(x)}^{\beta(x)} F(x\,,\ \xi)\, d\xi\right] = \int_{\alpha(x)}^{\beta(x)} \frac{\partial F}{\partial x} d\xi + F(x\,,\ \beta(x))\ \frac{d\beta(x)}{dx}\ - F(x\,,\ \alpha(x))\frac{d\alpha(x)}{dx}$$

In particular, we have

$$\frac{d}{dx}\left[\int_a^x K(x\ ,\ \xi)u(\xi)\,d\xi\right]=\int_a^x \frac{\partial K}{\partial x}u(\xi)\,d\xi+K(x\ ,\ x)\,u(x)\,.$$

6.5 SOLUTION OF BOUNDARY VALUE PROBLEMS

In differential equations, solving a boundary value issue for a given differential equation entails finding a solution to the differential equation under consideration of a given set of boundary conditions. When a boundary condition is prescribed, it specifies a number of different combinations of values of the unknown solution and its derivatives at different points.

Let I = (a, b) ⊆ R be an interval. Let p, q, r : (a, b) → R be continuous functions.

Throughout this chapter we consider the linear second order equation given by

$$y'' + p(x)y' + q(x)y = r(x), \quad a < x < b.$$

Corresponding to ODE (5.1), there are four important kinds of (linear) boundary conditions. They are given by

$$\begin{aligned}
\text{Dirichlet or First kind :} &\quad y(a) = \eta_1, \quad y(b) = \eta_2,\\
\text{Neumann or Second kind :} &\quad y'(a) = \eta_1, \quad y'(b) = \eta_2,\\
\text{Robin or Third or Mixed kind :} &\quad \alpha_1 y(a) + \alpha_2 y'(a) = \eta_1, \quad \beta_1 y(b) + \beta_2 y'(b) = \eta_2,\\
\text{Periodic :} &\quad y(a) = y(b), \quad y'(a) = y'(b).
\end{aligned}$$

Remark 5.1 (With regard to the periodic boundary condition) The coefficients of ODE (5.1) are periodic functions with period l=ba, and the solution of ODE (5.1) is defined by x = x + l. If x = x + l is also a solution of ODE (5.1), then the function x defined by x = l is likewise a solution of the equation. Assuming that a meets the periodic boundary conditions, the equations (a) = (a) and (a) = (a) and (a) = (a) (a). Given that there are no other solutions to IVP in this situation, it is necessary to conclude that Or, to put it another way, is a periodic function with period l. When compared to Initial value issues, Boundary value problems are more difficult to solve. For one thing, there are BVPs for which there are no solutions, and even if a solution does exist, there may be many more

BVPs. As a result, BVPs often fail to satisfy the requirements of existence and uniqueness. Using the following example, we can see all three alternatives.

Example:

Consider the equation

$$y'' + y = 0$$

(i) The BVP for equation (5.2) with boundary conditions y(0) = 1, y(π 2) = 1 has a unique solution. This solution is given by sin x + cos x.

(ii) The BVP for equation (5.2) with boundary conditions y(0) = 1, y(π) = 1 has no solutions.

(iii) The BVP for equation (5.2) with boundary conditions y(0) = 1, y(2π) = 1 has an infinite number of solutions.

6.6 EVALUATION OF DEFINITE INTEGRALS

Absolute integrals are distinguished by the presence of numbers written to the top and lower right of the integral symbol. This leaflet describes how to assess definite integrals and how to calculate their derivatives.

1. Definite integrals

The quantity

$$\int_a^b f(x)\,\mathrm{d}x$$

The definite integral of f(x) from a to b is referred to as the definite integral. The lower and higher boundaries of the integral are denoted by the integers a and b, respectively. If you want to see an example of how to assess definite integrals, examine the following one.

Example

Find $\int_1^4 x^2 \mathrm{d}x$.

Solution

First and foremost, the integration of x 2 is carried out in the conventional manner. However, in order to demonstrate that we are dealing with a definite integral, the result is often contained in square brackets, with the integration limits stated on the right side of the square bracket:

$$\int_1^4 x^2\,\mathrm{d}x = \left[\frac{x^3}{3} + c\right]_1^4$$

Then, the amount enclosed in square brackets is evaluated, first by letting x take the value of the higher limit, and then by letting x take the value of the lower limit, as shown in the following example. The value of the definite integral is found by calculating the difference between these two results:

$$\left[\frac{x^3}{3} + c\right]_1^4 = \text{(Evaluate at upper limit)} - \text{(evaluate at lower limit)}$$

$$\begin{aligned} &= \left(\frac{4^3}{3} + c\right) - \left(\frac{1^3}{3} + c\right) \\ &= \frac{64}{3} - \frac{1}{3} \\ &= 21 \end{aligned}$$

Note that the constants of integration cancel out. This will always happen, and so in future we can ignore them when we are evaluating definite integrals.

Example

Find $\int_{-2}^3 x^3 \mathrm{d}x$.

Solution

$$\begin{aligned}\int_{-2}^{3} x^3 \mathrm{d}x &= \left[\frac{x^4}{4}\right]_{-2}^{3} \\ &= \left(\frac{(3)^4}{4}\right) - \left(\frac{(-2)^4}{4}\right) \\ &= \frac{81}{4} - \frac{16}{4} \\ &= \frac{65}{4} \\ &= 16.25\end{aligned}$$

Example

Find $\int_0^{\pi/2} \cos x \, \mathrm{d}x$.

Solution

$$\begin{aligned}\int_0^{\pi/2} \cos x \, \mathrm{d}x &= [\sin x]_0^{\pi/2} \\ &= \sin\left(\frac{\pi}{2}\right) - \sin 0 \\ &= 1 - 0 \\ &= 1\end{aligned}$$

6.7 APPLICATION OF THE JOINED LAPLACE AND SUMUDU TRANSFORMS

There are various publications in the literature on the theory and applications of integral transforms, such as Laplace, Fourier, Mellin, and Hankel, to mention a few, but there is relatively little work on the power series transformation, such as the Sumudu transform, in the literature. This is most likely due to the fact that it is not commonly utilised yet. Watugala recently suggested the Sumudu transform, which may be found in. The features were first defined in and then used to partial differential equations; see, for example, for more discussion. In our research, we make use of the convolution notation as follows: a double convolution between two continuous functions F(x, y) and G(x, y) is defined as follows:

$$F_1(x, y) * *F_2(x, y) = \int_0^y \int_0^x F_1(x - \theta_1, y - \theta_2) F_2(\theta_1, \theta_2) d\theta_1 d\theta_2;$$

We refer to for further information on the double convolutions and derivatives, as well as their characteristics. The single Sumudu transform is defined over the set of all the functions in the set of functions

$$A = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{t/\tau_j}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}$$

By

$$F(u) = S\left[f(t); u\right] = \frac{1}{u} \int_0^\infty e^{-t/u} f(t)\, dt, \quad u \in (-\tau_1, \tau_2).$$

A necessary requirement for the existence of the Sumudu transform of a function f is of exponential order, which means that there are real constants in the function's domain.

$$M > 0,\ K_1, \text{ and } K_2, \text{ such that } |f(t, x)| \leq Me^{\frac{t}{K_1} + \frac{x}{K_2}}.$$

The Sumudu Transform

Theorem 6.1. If f is of exponential order, then its Sumudu transform S [f(t, x)] = F(v, u) exists and is given by

$$F(v,u)=\int_0^\infty\int_0^\infty e^{-\frac{t}{v}-\frac{x}{u}}f(t,x)dtdx,$$

where, $\frac{1}{u}=\frac{1}{\eta}+\frac{i}{\tau}$ *and* $\frac{1}{v}=\frac{1}{\mu}+\frac{i}{\xi}$*. The defining integral for* F *exists at points* $\frac{1}{u}+\frac{1}{v}=\frac{1}{\eta}+\frac{1}{\mu}+\frac{i}{\tau}++\frac{i}{\xi}$ *in the right half plane* $\frac{1}{\eta}+\frac{1}{\mu}>\frac{1}{K_1}+\frac{1}{K_2}$*.*

Proof. Using $\frac{1}{u}=\frac{1}{\eta}+\frac{i}{\tau}$ and $\frac{1}{v}=\frac{1}{\mu}+\frac{i}{\xi}$, we can express $F(v,u)$ as:

$$\begin{aligned}F(v,u) &= \int_0^\infty\int_0^\infty f(t,x)\cos\left(\frac{t}{\tau}+\frac{x}{\zeta}\right)e^{-\frac{t}{\eta}-\frac{x}{\mu}}dtdx\\ &\quad -i\int_0^\infty\int_0^\infty f(t,x)\sin\left(\frac{t}{\tau}+\frac{x}{\zeta}\right)e^{-\frac{t}{\eta}-\frac{x}{\mu}}dtdx.\end{aligned}$$

Then, for values of $\frac{1}{\eta}+\frac{1}{\mu}>\frac{1}{K_1}+\frac{1}{K_2}$, we have

$$\begin{aligned}&\int_0^\infty\int_0^\infty |f(t,x)|\left|\cos\left(\frac{t}{\tau}+\frac{x}{\zeta}\right)\right|e^{-\frac{t}{\eta}-\frac{x}{\mu}}dtdx\\ \leq\ & M\int_0^\infty\int_0^\infty e^{\left(\frac{1}{K_1}-\frac{1}{\eta}\right)t+\left(\frac{1}{K_2}-\frac{1}{\mu}\right)x}dtdx\\ \leq\ & M\left(\frac{\eta K_1}{\eta-K_1}\right)\left(\frac{\eta K_2}{\mu-K_2}\right)\end{aligned}$$

And

$$\begin{aligned}&\int_0^\infty\int_0^\infty |f(t,x)|\left|\sin\left(\frac{t}{\tau}+\frac{x}{\zeta}\right)\right|e^{-\frac{t}{\eta}-\frac{x}{\mu}}dtdx\\ \leq\ & M\int_0^\infty\int_0^\infty e^{\left(\frac{1}{K_1}-\frac{1}{\eta}\right)t+\left(\frac{1}{K_2}-\frac{1}{\mu}\right)x}dtdx\\ \leq\ & M\left(\frac{\eta K_1}{\eta-K_1}\right)\left(\frac{\eta K_2}{\mu-K_2}\right)\end{aligned}$$

which imply that the integrals defining the real and imaginary parts of F exist for value of Re($(\frac{1}{u}+\frac{1}{\mu})>\frac{1}{K_1}+\frac{1}{K_2}$, and this completes the proof.

Thus, we note that for a function f, the sufficient conditions for the existence of the Sumudu transform are to be piecewise continuous and of exponential order.

We also note that the double Sumudu transform of function f(t, x) is defined in [5], by

$$(2.1) \quad F(v,u) = S_2\left[f(t,x);(v,u)\right] = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-\left(\frac{t}{v}+\frac{x}{u}\right)} f(t,x)\,dt\,dx$$

S2 stands for double Sumudu transform and f(t, x) is a function that may be written as an infinite series with a convergent endpoint at the origin of the series. The derivative of convolution for two functions f and g is widely known at this point, and it is written as

$$\frac{d}{dx}(f*g)(x) = \frac{d}{dx}f(x)*g(x) \text{ or } f(x)*\frac{d}{dx}g(x)$$

And it can be easily proved that Sumudu transform is:

$$\frac{d}{dx}(f*g)(x) = \frac{d}{dx}f(x)*g(x) \text{ or } f(x)*\frac{d}{dx}g(x)$$

And it can be easily proved that Sumudu transform is:

$$\begin{aligned} S\left[\tfrac{d}{dx}(f*g)(x);\ v\right] &= uS\left[\tfrac{d}{dx}f(x);u\right] S\left[g(x);\ u\right] \\ & uS\left[f(x);\ u\right] S\left[\tfrac{d}{dx}g(x);\ u\right]. \end{aligned}$$

The double Sumudu and double Laplace transforms have strong relationships that may be expressed either as

$$\begin{aligned} &\text{(I)} \quad uvF(u,v) = \pounds_2\left(f(x,y);\left(\tfrac{1}{u},\tfrac{1}{v}\right)\right) \\ \text{or} \quad &\text{(II)} \quad psF(p,s) = \pounds_2\left(f(x,y);\left(\tfrac{1}{p},\tfrac{1}{s}\right)\right) \end{aligned}$$

where £2 represents the operation of double Laplace transform. In particular, the double Sumudu and double Laplace transforms interchange the image of sin(x + t) and cos(x + t). It turns out th

$$S_2\left[\sin(x+t)\right] = \mathcal{L}_2\left[\cos(x+t)\right] = \frac{u+v}{(1+u)^2(1+v)^2}$$

and $$S_2\left[\cos(x+t)\right] = \mathcal{L}_2\left[\sin(x+t)\right] = \frac{1}{(1+u)^2(1+v)^2}.$$

6.8 SUMUDU TRANSFORM IN CONTROL ENGINEERING

In the lack of beginning values as well as boundary conditions, a differential equation by itself is intrinsically under constrained. It is also generally known that a differential equation, together with its beginning values or boundary conditions, may be represented by an integral equation, and that it is feasible to solve the issue by employing this integral version of the differential equation. However, one of the most significant accomplishments and applications of integral transform techniques is the solution of second-order partial differential equations (PDEs). A new integral transform, known as the Sumudu transform, was recently introduced and used for partial derivatives of the Sumudu transform, which provided the complex inversion formula in order to solve differential equations in various applications of system engineering, control theory, and applied physics. According to Asiru in the convolution theorem of the Sumudu transform may be established]. Several applications of this novel transform have been shown, including the solution of ordinary differential equations and control engineering issues. Some essential features of the Sumudu transform were developed in which may be found here. This novel transform was applied to the one-dimensional neutron transport equation in and the results were published. It was really examined in how the link between the double Sumudu and the double Laplace transformations works. A further extension of the Sumudu transform was made to distributions in the chapter and some of their features were also investigated. As a result, there have been several works on the Sumudu transform that have been applied to a variety of issues.

In this chapter, we establish the Sumudu transform of convolution for matrices and apply it to solve a regular system of differential equations with a constant coefficient of convergence.

Throughout the chapter we use a square matrix, , $P = [P_{ij}]$ of regular system having size $n \times n$ of polynomials and the associated determinant, , detP. If detPis not the zero polynomial which we write as det $(P) \neq 0)$, we have degdet $(P)] \leq |N(P)|$ where N(is the degree of the polynomials in the regular matrix P. The case of equality is so important that we make the following statement. We say that P is regular if detP/ 0 and the condition

$$\deg[\det(P)] = |N(P)|,$$

Where NjP considered as the highest power of the variable term that occurs in the jth column of matrix P, that is,

$$N_j(P) = \max_{\substack{1\le i\le m \\ P_{ij}\neq 0}} [\deg([P_{ij}])].$$

Next we extend the result given in 4 as follows. For each i and j we define Ψi,j P xto be the 1 × Nj matrix of polynomials given by the matrix product

$$\Psi_P^{(i,j)}(x) = \left(\frac{1}{x}\frac{1}{x^2}\frac{1}{x^3}\cdots\frac{1}{x^{N-1}}\right)\begin{pmatrix} a_1 & a_2 & . & . & . & a_{N_j} \\ a_2 & a_3 & . & . & a_{N_j} & 0 \\ a_3 & . & . & a_{N_j} & 0 & 0 \\ . & . & . & . & . & . \\ a_{N_j} & 0 & . & . & . & 0 \end{pmatrix},$$

Where the ak are the coefficients of P_{ij} and $P_{ij} = \sum_{k=0}^{N_j} a_k/x^k$. In terms of 7.1in 4we have

$$\Psi_P^{(i,j)}(x) = \left(\Psi_{P_{ij}}(x)\, 0\ 0\cdots 0\right),$$

Where the number of zero indicated is Nj−degPijWe define ΨP to be matrix of polynomials and having size m × |N| defined in terms of the array of matrices:

$$\Psi_P = \begin{pmatrix} \Psi_P^{(1,1)} & \Psi_P^{(1,2)} & \cdots & \Psi_P^{(1,n)} \\ \Psi_P^{(2,1)} & \Psi_P^{(2,2)} & \cdots & \Psi_P^{(2,n)} \\ \Psi_P^{(3,1)} & . & \cdots & \Psi_P^{(3,n)} \\ . & . & \cdots & . \\ \Psi_P^{(m,1)} & . & \cdots & \Psi_P^{(m,n)} \end{pmatrix}.$$

For each complex number $x, \Psi_P(x)$ define a linear mapping of $\mathbf{C}^{|N|}$ into $\mathbf{C}^{m}$. If any Nj is zero, $\Psi_P^{(i,j)}$ is the empty matrix for all i and the corresponding column of matrices in Ψ_P is absent.

If $N_j = 0$ for al j, $\Psi_P(x)$ is defined to be the unique linear mapping of {0} C0 into $\mathbf{C}^m$; its matrix representation is then the empty matrix. In particular consider

$$\Psi_P(x) = \begin{pmatrix} x^4 + 2x^2 & 1 - 2x \\ 3x^4 + 2x & 4x^3 \end{pmatrix}.$$

Then we have N1= 4, N2 = 3, and N = 4, 3. Thus ΨP is the 3 × 7 matrix computed as

$$\Psi_P(x) = \begin{pmatrix} \left(\frac{1}{x}\ \frac{1}{x^2}\ \frac{1}{x^3}\ \frac{1}{x^4}\right)\begin{pmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} & \left(\frac{1}{x}\ \frac{1}{x^2}\ \frac{1}{x^3}\right)\begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \left(\frac{1}{x}\ \frac{1}{x^2}\ \frac{1}{x^3}\ \frac{1}{x^4}\right)\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} & \left(\frac{1}{x}\ \frac{1}{x^2}\ \frac{1}{x^3}\right)\begin{pmatrix} 0 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{x^2} + \frac{1}{x^4} & \frac{2}{x} + \frac{1}{x^3} & \frac{1}{x^2} & \frac{1}{x} & -\frac{2}{x} & 0 & 0 \\ \frac{2}{x} + \frac{1}{x^4} & \frac{1}{x^3} & \frac{1}{x^2} & \frac{1}{x} & \frac{4}{x^3} & \frac{4}{x^2} & \frac{4}{x} \end{pmatrix}.$$

In general, if $f = (f_1, f_2, \ldots, f_p)$ is a sequence of functions on (a, b), $h = h_1, h_2, \ldots, h_p$ with each h being an integer ≥ 0 and fi being hi 1 times differentiable on a, b, we will write, using the notation in 4,

$$\Phi(f, a; h) = (\Phi(f_1, a; h_1), \Phi(f_1, a; h_2), \ldots, \Phi(f_p, a; h_p)) \in \mathbf{C}^{|h|},$$
$$F(f, b; h) = (F(f_1, b; h_1), F(f_1, b; h_2), \ldots, F(f_p, b; h_p)) \in \mathbf{C}^{|h|},$$

CHAPTER 7

CONCLUSION

The definition and application of the new complex SEE integral transform to solution of ordinary differential equations has been demonstrated In the present chapter, a new integral transform namely SEE transform was applied to solve linear ordinary and partial differential equations with constant coefficients. Also, its applicability demonstrated using different partial differential equations (wave, heat, Laplace), we find the particular solutions of these equations. This part of the course introduces two extremely powerful methods to solving differential equations: the Fourier and the Laplace transforms. Beside its practical use, the Fourier transform is also of fundamental importance in quantum mechanics, providing the correspondence between the position and momentum representations of the Heisenberg commutation relations. An integral transform is useful if it allows one to turn a complicated problem into a simpler one. The transforms we will be studying in this part of the course are mostly useful to solve differential and, to a lesser extent, integral equations. The idea behind a transform is very simple. To be definite suppose that we want to solve a differential equation, with unknown function f. One first applies the transform to the differential equation to turn it into an equation one can solve easily: often an algebraic equation for the transform F of f.

Through the course of this chapter, the symbols C (for complex numbers), R (for real numbers), N (for non-negative integers), Z (for real numbers), and N0 (for non-negative integers) will refer to the sets of all complex numbers, real numbers, integers, natural numbers, and non-negative integers, respectively. The Sumudu transform has already shown great promise, owing to its straightforward construction and the unusual and valuable qualities that result as a result. It has been shown here and elsewhere that it may be used to assist in the solution of complex issues in engineering mathematics and applied science. Despite the promise offered by this new operator, only a few theoretical studies have been published in the literature during a fifteen-year period, despite the possibility offered by this new operator. The Sumudu transform is not mentioned in most, if not all, of the transform theory texts that are currently accessible. No mention of the Sumudu transform can be found in any of the more recent well-known comprehensive handbooks, including. Perhaps this is due to the fact that no transform with this name (in the traditional sense) was announced until the late 1980s and early 1990s of the previous century. On the other hand, it is important to note that an analogous formulation, known as the s-multiplied Laplace transform, was disclosed as early as 1948 (see, for example, and references), if not before.